Linear Quadratic Gaussian Control Design with Loop Transfer Recovery

Leonid Freidovich

Department of Mathematics
Michigan State University
MI 48824, USA

e-mail: leonid@math.msu.edu
http://www.math.msu.edu/~leonid/

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Outline
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(a) Linear quadratic regulation [LQR]
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(b) $\mathcal{H}_2$ design [LQG]
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(c) Idea of loop transfer recovery [LTR]
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(b) $\mathcal{H}_2$ design [LQG]

(c) Idea of loop transfer recovery [LTR]

(d) Technical implementations of LQR/LTR
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(b) $\mathcal{H}_2$ design [LQG]

(c) Idea of loop transfer recovery [LTR]

(d) Technical implementations of LQR/LTR

(e) References
Linear Quadratic Regulation [LQR]

\[ \dot{x} = Ax + B_2 u, \]
\[ x(0) = x_0. \]  

- \((A, B_2)\) is controllable.
Linear Quadratic Regulation [LQR]

\[ \dot{x} = Ax + B_2u, \]
\[ x(0) = x_0. \] (1)

- \((A, B_2)\) is controllable.

**Goal:** Design state feedback controller

\[ u = K(s)x, \] \[ K(s) \text{ is a proper real rational transfer function} \] (2)

depending on \(A\) and \(B_2\) such that for arbitrary \(x_0:\)

1. the closed-loop system (1), (2) is internally stable,

2. the solution \(x(t)\) and the control signal \(u(t)\) satisfy certain specifications.

[LQR/LQG/LTR]
Let us choose $C_1$, $D_{12}$ and define performance index:

$$J_{lqr} = \|z\|_2^2 = \|C_1x + D_{12}u\|_2^2 \Delta \int_0^\infty [C_1x(t) + D_{12}u(t)]*[C_1x(t) + D_{12}u(t)]dt \quad (3)$$
Let us choose $C_1, \ D_{12}$ and define performance index:

$$J_{lqr} = \|z\|^2 = \|C_1 x + D_{12} u\|^2 \overset{\Delta}{=} \int_{0}^{\infty} [C_1 x(t) + D_{12} u(t)]^* [C_1 x(t) + D_{12} u(t)] \, dt \quad (3)$$

If $C_1^* D_{12} = 0$ then $J_{lqr} = \|C_1 x\|^2 + \|D_{12} u\|^2$. 
Let us choose $C_1, D_{12}$ and define performance index:

$$J_{lqr} = \|z\|_2^2 = \|C_1 x + D_{12} u\|_2^2 \overset{\Delta}{=} \int_0^\infty [C_1 x(t) + D_{12} u(t)]^* [C_1 x(t) + D_{12} u(t)] \, dt \quad (3)$$

If $C_1^* D_{12} = 0$ then $J_{lqr} = \|C_1 x\|_2^2 + \|D_{12} u\|_2^2$.

**LQR problem:** Design state feedback controller (2) depending on $A$, $B_2$, $C_1$, and $D_{12}$ such that for arbitrary $x_0$:

1. the closed-loop system (1), (2) is internally stable,

2. the performance index (3) is the smallest.
The solution exists if

- $D_{12}^* D_{12} > 0,$
- $(C_1, A)$ is detectable;
- $egin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\omega.$
Properties of the solution:

1. optimal controller $K(s) = K_{lqr}$ is the constant gain:

$$K_{lqr} = -(D_{12}^*D_{12})^{-1}(B_2^*X + D_{12}^*C_1),$$

where $X \geq 0$ is the stabilizing solution of the Riccati equation:

$$\bar{A}^*X + X\bar{A} - XB_2(D_{12}^*D_{12})^{-1}B_2^*X + C_1^*[I - D_{12}(D_{12}^*D_{12})^{-1}D_{12}^*]C_1 = 0,$$

with $\bar{A} = A - B_2(D_{12}^*D_{12})^{-1}D_{12}^*C_1$. 
Properties of the solution:

1. optimal controller \( K(s) = K_{lq} \) is the constant gain:

\[
K_{lq} = -(D_{12}^*D_{12})^{-1}(B_2^*X + D_{12}^*C_1),
\]

where \( X \geq 0 \) is the stabilizing solution of the Riccati equation:

\[
\bar{A}^*X + X\bar{A} - XB_2(D_{12}^*D_{12})^{-1}B_2^*X + C_1^*[I - D_{12}(D_{12}^*D_{12})^{-1}D_{12}^*]C_1 = 0,
\]

with \( \bar{A} = A - B_2(D_{12}^*D_{12})^{-1}D_{12}^*C_1 \),

2. \( A + B_2K_{lq} \) is Hurwitz and \( \lim_{t \to \infty} x(t) = 0 \),
Properties of the solution:

1. optimal controller \( K(s) = K_{lqr} \) is the constant gain:

\[
K_{lqr} = -(D_{12}^* D_{12})^{-1}(B_2^* X + D_{12}^* C_1),
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where \( X \geq 0 \) is the stabilizing solution of the Riccati equation:

\[
\bar{A}^* X + X \bar{A} - X B_2 (D_{12}^* D_{12})^{-1} B_2^* X + C_1^*[I - D_{12} (D_{12}^* D_{12})^{-1} D_{12}^*] C_1 = 0,
\]

with \( \bar{A} = A - B_2 (D_{12}^* D_{12})^{-1} D_{12}^* C_1 \),

2. \( A + B_2 K_{lqr} \) is Hurwitz and \( \lim_{t \to \infty} x(t) = 0 \),

3. guaranteed 6 dB \( (= 20 \log 2 \) gain margin and \( 60^o \) phase margin in both directions.

[LQR/LQG/LTR]
Linear Quadratic Gaussian [LQG or \( H_2 \)] problem

Model:

\[
\begin{align*}
\dot{x} &= Ax + B_2 u, \\
y &= C_2 x + D_{21} u, \\
x(0) &= x_0.
\end{align*}
\]

\[\text{(4)}\]

Assumptions:

- \((A, B_2)\) is controllable,
- \((C_2, A)\) is detectable.
Goal: Design output feedback controller

\[ u = K(s)y, \quad [K(s) \text{ is a proper real rational transfer function}] \quad (5) \]

depending on \( A, B_2, C_2, \) and \( D_{21} \) such that for arbitrary \( x_0 \):

1. the closed-loop system (6), (5) is internally stable,

2. the solution \( x(t) \) and the control signal \( u(t) \) satisfy certain specifications.
Goal: Design output feedback controller

\[ u = K(s)y, \quad [K(s) \text{ is a proper real rational transfer function}] \quad (5) \]

depending on \( A, B_2, C_2, \text{ and } D_{21} \) such that for arbitrary \( x_0 \):

1. the closed-loop system (6), (5) is internally stable,

2. the solution \( x(t) \) and the control signal \( u(t) \) satisfy certain specifications.

Let us choose \( C_1, D_{12} \), as before.
Goal: Design output feedback controller

\[ u = K(s)y, \quad [K(s) \text{ is a proper real rational transfer function}] \quad (5) \]

depending on \( A, B_2, C_2, \text{ and } D_{21} \) such that for arbitrary \( x_0 \):

1. the closed-loop system (6), (5) is internally stable,

2. the solution \( x(t) \) and the control signal \( u(t) \) satisfy certain specifications.

Let us choose \( C_1, D_{12} \), as before. It could be shown that if \( D_{12} \neq 0 \), then the output controller minimizing \( J_{lqr} = \|z\|^2 = \|C_1x + D_{12}u\|_2^2 \) cannot be proper (i.e. solution does not exist).
Consider

\[ \dot{x} = Ax + B_1 w + B_2 u, \]
\[ y = C_2 x + D_{21} u, \]
\[ x(0) = x_0. \]

(6)
Consider

\[ \dot{x} = Ax + B_1w + B_2u, \]
\[ y = C_2x + D_{21}u, \]
\[ x(0) = x_0. \]  \hspace{1cm} (6)

Define performance index

\[ J_{lqg} = \left\| T_{w \rightarrow z} \right\|_2^2 = \max_{\|w\|_2 \leq 1} \left\{ \frac{\|z\|_\infty^2}{\|w\|_2^2} \right\} = \max_{\|w\|_2 \leq 1} \left\{ \frac{\|C_1x + D_{12}u\|_\infty^2}{\|w\|_2^2} \right\}. \]  \hspace{1cm} (7)
Consider

\[ \dot{x} = Ax + B_1 w + B_2 u, \]
\[ y = C_2 x + D_{21} u, \]
\[ x(0) = x_0. \]  

(6)

Define performance index

\[ J_{lqg} = \|Tw \rightarrow z\|_2^2 = \max_{\|w\|_2 \leq 1} \left\{ \frac{\|z\|_\infty^2}{\|w\|_2^2} \right\} = \max_{\|w\|_2 \leq 1} \left\{ \frac{\|C_1 x + D_{12} u\|_\infty^2}{\|w\|_2^2} \right\}. \]  

(7)

LQG problem: Design output feedback controller (5) depending on \( A, \ B_1, \ B_2, \ C_1, \ C_2, \ D_{12}, \ \text{and} \ D_{21} \) such that for arbitrary \( x_0 \):

1. the closed-loop system (6), (5) is internally stable,

2. the performance index (7) is the smallest.
The solution exists if

- \( D_{12}^* D_{12} > 0 \),
- \[
\begin{bmatrix}
A - j\omega I & B_2 \\
C_1 & D_{12}
\end{bmatrix}
\]
  has full column rank for all \( \omega \).
- \( D_{21} D_{21}^* > 0 \),
- \[
\begin{bmatrix}
A - j\omega I & B_1 \\
C_2 & D_{21}
\end{bmatrix}
\]
  has full row rank for all \( \omega \).
Properties of the solution:

1. optimal $K(s)$ is the observer based controller:

$$
\begin{align*}
  u &= K_{lqg} \hat{x}, \\
  \dot{\hat{x}} &= A \hat{x} + B_2 u + L_{lqg} (C_2 \hat{x} + D_{21} u - y), \\
  \hat{x}(0) &= \hat{x}_0,
\end{align*}
$$
Properties of the solution:

1. optimal $K(s)$ is the observer based controller:

$$u = K_{lqg} \hat{x},$$
$$\dot{\hat{x}} = A\hat{x} + B_2 u + L_{lqg}(C_2 \hat{x} + D_2 u - y),$$
$$\hat{x}(0) = \hat{x}_0,$$

where

$$K_{lqg} = K_{lqr}$$

and

$$L_{lqg} = -(Y C_2^* + B_1 D_2^*) (D_2 D_2^*)^{-1},$$

where $Y \geq 0$ is the stabilizing solution of the Riccati equation:

$$A^* Y + Y A - Y C_2^* (D_2 D_2^*)^{-1} C_2 Y + B_1 [I - D_2^* (D_2 D_2^*)^{-1} D_2] B_1^* = 0,$$

with $A = A - B_1 D_2^* (D_2 D_2^*)^{-1} C_2,$
2. observer cannot be too fast since transient with peaking is not ‘compatible’ with small values of $\|C_1 x + D_{12} u\|_\infty$, 
2. observer cannot be too fast since transient with peaking is not ‘compatible’ with small values of $\|C_1x + D_{12}u\|_\infty$,

3. if $w(t) \equiv 0$ then $\lim_{t \to \infty} x(t) = 0$, 
2. observer cannot be too fast since transient with peaking is not ‘compatible’ with small values of $\|C_1 x + D_{12} u\|_\infty$,

3. if $w(t) \equiv 0$ then $\lim_{t \to \infty} x(t) = 0$,

4. in general, neither gain nor phase margins are guaranteed (have to check robustness for each particular design).
Consider the observer based controller for the system (4) with $D_{21} = 0$

\[
\begin{align*}
\dot{x} &= Ax + B_2 u, \\
y &= C_2 x, \\
x(0) &= x_0,
\end{align*}
\]
Consider the observer based controller for the system (4) with \( D_{21} = 0 \):

\[
\begin{align*}
\dot{x} &= Ax + B_2 u, \\
y &= C_2 x, \\
x(0) &= x_0, \\
\end{align*}
\]

\[
\begin{align*}
u &= K_{lqr} \hat{x}, \\
\dot{\hat{x}} &= A \hat{x} + B_2 u + L(C_2 \hat{x} - y), \\
\hat{x}(0) &= \hat{x}_0.
\end{align*}
\]

Assumptions:

- \((A, B_2)\) is controllable,
- \((C_2, A)\) is detectable,
- \(K_{lqr}\) is the optimal gain from the LQR problem.
Goal: Design the observer gain $L$ so that the closed-loop system (8) recovers internal stability and some of the robustness properties (gain and phase margins) of the LQR design.
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Compare

$$\dot{x} = Ax + B_2u$$

Figure 1: $\hat{u} = \{L_t(s)\} u$

$$\hat{u} = K_{lqr} (sI - A)^{-1} B_2u \overset{def}{=} \{L_t(s)\} u$$
and

\[ \hat{u} = K_{\text{lqr}} \left[ -(sI - A - LC_2 - B_2 K_{\text{lqr}})^{-1}LC_2 \right] (sI - A)^{-1} B_2 u \overset{\text{def}}{=} \{ L_0(s) \} u \]

observer: \[ \dot{x} = A\hat{x} + B_2 [K_{\text{lqr}}\hat{x}] + LC_2 (\hat{x} - x) \]

\[ \dot{x} = Ax + B_2 u \]

Figure 2: \( \hat{u} = \{ L_0(s) \} u \)
and

\[ \dot{u} = K_{lqr} \left[ -(sI - A - LC_2 - B_2K_{lqr})^{-1}LC_2 \right] (sI - A)^{-1}B_2u \]

\[ \text{observer:} \quad \dot{x} = A\hat{x} + B_2[K_{lqr}\hat{x}] + LC_2(\hat{x} - x) \]

\[ \dot{x} = Ax + B_2u \]

**Figure 2:** \( \hat{u} = \{L_o(s)\} u \)

We want \( L_o(s) \equiv L_t(s) \), but \( L_o(s) \) is biproper and \( L_t(s) \) is strictly proper.

[LQR/LQG/LTR]
LQG/LTR problem: Design the observer gain $L$ so that

1. the closed-loop system (8) is internally stable,

2. for all $\omega \in [0, \omega_{max}]$:

$$L_o(j\omega) \approx L_t(j\omega),$$

where $\omega_{max} \gg 1$,

**target (under state feedback) open-loop transfer function is**

$$L_t(s) = K_{lqr}(sI - A)^{-1}B_2,$$

**achieved (under output feedback) open-loop transfer function is**

$$L_o(s) = K_{lqr}T_{x\rightarrow\hat{x}}(s)(sI - A)^{-1}B_2,$$

and ‘excited’ observer transfer function is

$$T_{x\rightarrow\hat{x}}(s) = [-(sI - A - LC_2 - B_2K_{lqr})^{-1}LC_2].$$
It could be shown that if

\[-L[I - C_2(j\omega I - A)^{-1}L]^{-1} = B_2[C_2(j\omega I - A)^{-1}B_2]^{-1}\]

then

\[L_o(j\omega) = L_t(j\omega).\]
Doyle – Stein formula

It could be shown that if

\[-L[I - C_2(jωI - A)^{-1}L]^{-1} = B_2[C_2(jωI - A)^{-1}B_2]^{-1}\]

then

\[L_o(jω) = L_t(jω).\]

Suppose

\[\lim_{ε→0}\{εL(ε)\} = -B_2W,\]

where \(W\) is not singular and \(ε \ll 1.\)
Doyle – Stein formula

It could be shown that if

\[-L[I - C_2(j\omega I - A)^{-1}L]^{-1} = B_2[C_2(j\omega I - A)^{-1}B_2]^{-1}\]

then

\[L_o(j\omega) = L_t(j\omega).\]

Suppose

\[\lim_{\varepsilon \to 0} \{\varepsilon L(\varepsilon)\} = -B_2 W,\]

where \(W\) is not singular and \(\varepsilon \ll 1\). Then

\[\lim_{\varepsilon \to 0} \{B_2 W[\varepsilon I + C_2(j\omega I - A)^{-1}B_2 W]^{-1}\} = B_2[C_2(j\omega I - A)^{-1}B_2]^{-1}\]

and hence

\[\lim_{\varepsilon \to 0} \{L_o(j\omega)\} = L_t(j\omega).\]
Remarks on \((-L(\varepsilon)[I - C_2(j\omega I - A)^{-1}L(\varepsilon)]^{-1} \approx B_2[C_2(j\omega I - A)^{-1}B_2]^{-1}\))

- In general, the choice \(L(\varepsilon) = -B_2W/\varepsilon\) does not guarantee internal stability of the closed-loop system.
Remarks on \((-L(\varepsilon)[I - C_2(j\omega I - A)^{-1}L(\varepsilon)]^{-1} \approx B_2[C_2(j\omega I - A)^{-1}B_2]^{-1}\))

- In general, the choice \(L(\varepsilon) = -B_2W/\varepsilon\) does not guarantee internal stability of the closed-loop system.
- Since

\[-L(\varepsilon)[I - C_2(j\omega I - A)^{-1}L(\varepsilon)]^{-1} = -(j\omega I - A)(j\omega I - [A + L(\varepsilon)C_2])^{-1}L(\varepsilon),\]
Remarks on  
\[-L(\varepsilon)[I - C_2(j\omega I - A)^{-1}L(\varepsilon)]^{-1} \approx B_2[C_2(j\omega I - A)^{-1}B_2]^{-1}\]

- In general, the choice  \(L(\varepsilon) = -B_2W/\varepsilon\) does not guarantee internal stability of the closed-loop system.

- Since

\[-L(\varepsilon)[I - C_2(j\omega I - A)^{-1}L(\varepsilon)]^{-1} = -(j\omega I - A)(j\omega I - [A + L(\varepsilon)C_2])^{-1}L(\varepsilon),\]

if  \(\varepsilon L(\varepsilon) = -B_2W + O(\varepsilon)\), then some of the eigenvalues of  \([A + L(\varepsilon)C_2]\) (i.e. poles of the observer error dynamics) approach the zeros of  \([C_2(j\omega I - A)^{-1}B_2]\) (i.e. zeros of the plant) and the others approach infinity as  \(\varepsilon \to 0\).
Remarks on \((-L(\varepsilon)[I - C_2(j\omega I - A)^{-1}L(\varepsilon)]^{-1} \approx B_2[C_2(j\omega I - A)^{-1}B_2]^{-1}\))

- In general, the choice \(L(\varepsilon) = -B_2 W/\varepsilon\) does not guarantee internal stability of the closed-loop system.

- Since

\[-L(\varepsilon)[I - C_2(j\omega I - A)^{-1}L(\varepsilon)]^{-1} = -(j\omega I - A)(j\omega I - [A + L(\varepsilon)C_2])^{-1}L(\varepsilon),\]

if \(\varepsilon L(\varepsilon) = -B_2 W + O(\varepsilon),\) then some of the eigenvalues of \([A + L(\varepsilon)C_2]\) (i.e. poles of the observer error dynamics) approach the zeros of \([C_2(j\omega I - A)^{-1}B_2]\) (i.e. zeros of the plant) and the others approach infinity as \(\varepsilon \to 0\).

- If the plant is nonminimum-phase then \(A + L(\varepsilon)C_2\) is not Hurwitz for small values of \(\varepsilon\).
Remarks on \((-L(\varepsilon)[I - C_2(j\omega I - A)^{-1}L(\varepsilon)]^{-1} \approx B_2[C_2(j\omega I - A)^{-1}B_2]^{-1}\))

- In general, the choice \(L(\varepsilon) = -B_2W/\varepsilon\) does not guarantee internal stability of the closed-loop system.

- Since

\[-L(\varepsilon)[I - C_2(j\omega I - A)^{-1}L(\varepsilon)]^{-1} = -(j\omega I - A)(j\omega I - [A + L(\varepsilon)C_2])^{-1}L(\varepsilon),\]

if \(\varepsilon L(\varepsilon) = -B_2W + O(\varepsilon),\) then some of the eigenvalues of \([A + L(\varepsilon)C_2]\) (i.e. poles of the observer error dynamics) approach the zeros of \([C_2(j\omega I - A)^{-1}B_2]\) (i.e. zeros of the plant) and the others approach infinity as \(\varepsilon \to 0.\)

- If the plant is nonminimum-phase then \(A + L(\varepsilon)C_2\) is not Hurwitz for small values of \(\varepsilon.\)

- Some of the observer’s modes are fast. Therefore peaking phenomenon must occur.
LQR design of \( L(\varepsilon) \)

Consider LQR problem for the system

\[
\begin{align*}
\dot{x} &= \bar{A}x + \bar{B}_2 \tilde{u}, \\
\bar{x}(0) &= \bar{x}_0,
\end{align*}
\]

where \( \bar{A} = A^* \) and \( \bar{B}_2 = C_2^* \).
LQR design of $L(\varepsilon)$

Consider LQR problem for the system

$$\dot{x} = \bar{A}\bar{x} + \bar{B}_2\bar{u},$$
$$\bar{x}(0) = \bar{x}_0,$$

where $\bar{A} = A^*$ and $\bar{B}_2 = C_2^*$.

Take $\bar{C}_1(\varepsilon) = B_2^*/\varepsilon$ (cheap control)
LQR design of $L(\varepsilon)$

Consider LQR problem for the system

\[
\dot{x} = \bar{A} \bar{x} + \bar{B}_2 \bar{u}, \\
\bar{x}(0) = \bar{x}_0,
\]

where $\bar{A} = A^*$ and $\bar{B}_2 = C_2^*$.

Take $\bar{C}_1(\varepsilon) = B_2^*/\varepsilon$ (cheap control) and $\bar{D}_{12}$ independent of $\varepsilon$ so that $\bar{D}_{12}^* \bar{D}_{12} = R > 0$ and $\bar{C}_1^* \bar{D}_{12} = 0$. 
Consider LQR problem for the system

\[ \dot{x} = \bar{A} \bar{x} + \bar{B}_2 \bar{u}, \]
\[ \bar{x}(0) = \bar{x}_0, \]

where \( \bar{A} = A^* \) and \( \bar{B}_2 = C^* \).

Take \( \bar{C}_1(\epsilon) = B_2^*/\epsilon \) (cheap control) and \( \bar{D}_{12} \) independent of \( \epsilon \) so that \( \bar{D}_{12}^* \bar{D}_{12} = R > 0 \) and \( \bar{C}_1^* \bar{D}_{12} = 0 \).

Suppose \( \bar{K}_{lq}(\epsilon) \) is the solution gain of the problem above for each \( \epsilon > 0 \).

Take

\[ L(\epsilon) = \left[ \bar{K}_{lq}(\epsilon) \right]^*. \]
Then

\[
\left[ \bar{A} + \bar{B}_2 \bar{K}_{lqr}(\varepsilon) \right]^* = A + L(\varepsilon)C_2
\]

is Hurwitz
Then

\[
[\bar{A} + \bar{B}_2 \bar{K}_{lqr}(\varepsilon)]^* = A + L(\varepsilon)C_2
\]

is Hurwitz and

\[
L(\varepsilon) = -\bar{X}(\varepsilon)C_2^*R^{-1},
\]

where \( \bar{X}(\varepsilon) \) is the stabilizing solution of the Riccati equation

\[
A\bar{X}(\varepsilon) + \bar{X}(\varepsilon)A^* - \bar{X}(\varepsilon)C_2^*R^{-1}C_2\bar{X}(\varepsilon) + B_2B_2^*/\varepsilon^2 = 0.
\]
Then

\[ [\bar{A} + \bar{B}_2 \bar{K}_{lq}(\varepsilon)]^* = A + L(\varepsilon)C_2 \]

is Hurwitz and

\[ L(\varepsilon) = -\bar{X}(\varepsilon)C_2^*R^{-1}, \]

where \( \bar{X}(\varepsilon) \) is the stabilizing solution of the Riccati equation

\[ A\bar{X}(\varepsilon) + \bar{X}(\varepsilon)A^* - \bar{X}(\varepsilon)C_2^*R^{-1}C_2\bar{X}(\varepsilon) + B_2B_2^*/\varepsilon^2 = 0. \]

It could be shown that if the plant is of minimum phase (and left invertible) then

\[ \lim_{\varepsilon \to 0} \{\varepsilon^2[A\bar{X}(\varepsilon) + \bar{X}(\varepsilon)A^*]\} = 0 \]

and

\[ \lim_{\varepsilon \to 0} \{\varepsilon L(\varepsilon)\} = \lim_{\varepsilon \to 0} \{-\varepsilon\bar{X}(\varepsilon)C_2^*R^{-1}\} = -B_2R^{-1/2} = -B_2W \]
Alternative designs of $L(\varepsilon)$

For the ‘broken’ closed-loop system

\[
\begin{align*}
\dot{x} &= Ax + B_2 u, \\
y &= C_2 x + D_{21} u, \\
x(0) &= x_0,
\end{align*}
\]
Alternative designs of $L(\varepsilon)$

For the ‘broken’ closed-loop system

\[ \dot{x} = Ax + B_2u, \]
\[ y = C_2x + D_{21}u, \]
\[ x(0) = x_0, \]
\[ \hat{u} = K_{lqr}\hat{x}, \]
\[ \dot{\hat{x}} = A\hat{x} + B_2\hat{u} + L(C_2\hat{x} + D_{21}\hat{u} - y), \]
\[ \hat{x}(0) = \hat{x}_0. \]

we have: \[ L_t(s) = K_{lqr}(sI - A)^{-1}B_2 \quad \text{and} \]

\[ L_o(s) = K_{lqr}[-(sI - A - LC_2 - B_2K_{lqr} + LD_{21}K_{lqr})^{-1}]L[C_2(sI - A)^{-1}B_2 + D_{21}]. \]

[LQR/LQG/LTR]
It could be shown that

\[ L_t(s) - L_o(s) = M(s)[I + M(s)]^{-1}(I - K_{lqr}B_2), \]

where \( M(s) = -K_{lqr}([sI - A]^{-1} - LC_2)^{-1}(B_2 + LD_{21}) \).

Therefore it is reasonable to search for \( L \) that minimizes \( \|M(s)\|_2 \) or \( \|M(s)\|_\infty \).

Also, a pole-placement technique can be used to design the observer gain as well. However, it is not trivial and the system need to be transformed into the special coordinate basis.
References


[LQR/LQG/LTR]


