Virtual-Constraints-Based Design of Stable Oscillations of Furuta Pendulum: Theory and Experiment
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Introduction and Problem formulation
Generating and stabilizing periodic orbits for underactuated mechanical systems is a challenging task both for developing algorithms and for their implementation suitable for experiments. The subject of investigation in this paper is the Furuta pendulum, see Figure 1, which is a two-degree-of-freedom mechanical system, widely used for research, as well as for experimental and educational purposes.

\[ M(q)\dot{q}+C(q, \dot{q})\dot{q}+G(q) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tau, \quad q = \begin{bmatrix} \phi \\ \theta \end{bmatrix} \] (1)

where

\[ M = \begin{bmatrix} \alpha + \beta \sin^2 \theta & \gamma \cos \theta \\ \gamma \cos \theta & \beta \end{bmatrix}, \quad C = \begin{bmatrix} \beta \cos \theta \sin \theta \phi & \beta \cos \theta \sin \phi - \gamma \sin \theta \phi \\ -\beta \cos \theta \sin \theta \phi & \beta \cos \theta \sin \phi + \gamma \sin \theta \phi \end{bmatrix} \]

Identified lumped parameters:
\[ \alpha \approx 3.7 \cdot 10^{-3}, \quad \beta \approx 1.4 \cdot 10^{-3}, \quad \gamma \approx 0.9077 \cdot 10^{-3}, \quad \delta \approx 59.37 \cdot 10^{-3}. \]

![Figure 1. (Left) The Furuta pendulum used in the experiments, built at the Dept. of Applied Physics and Electronics, Umeå University. (Right) Equations of motion. Here \( \phi \) is the angle of the arm and \( \theta \) is the angle of the pendulum, measured from the upright position.](image)

The main contribution of this paper is a detailed description of a procedure for feedback controller design that ensures existence of a non-trivial periodic solution of the closed-loop dynamics of the Furuta pendulum and exponential orbital stability of the corresponding artificially shaped periodic trajectory. The feedback control design is based on a general approach, developed in [2] and extended here. We also discuss details and challenges that must be faced in order to make the approach work, not only in simulations, but for a real hardware set-up.

Outline of Design Procedure (details in the paper)

**Step 1. Choice of Constraint.**
To generate oscillations of the Furuta pendulum, we have chosen a particular virtual holonomic constraint

\[ \phi = K \theta + \theta_0, \] (2)

where \( \phi \) is the angle of the pendulum, measured from the upright position.

**Step 2. Computation of Reduced Dynamics.**
Compute the dynamics of the Furuta pendulum (1) projected to the virtual holonomic constraint (2). The reduced dynamics take the form

\[ (\beta + \gamma K \cos \theta) \ddot{\theta} - \beta K^2 \ddot{\phi} \sin \theta \cos \theta - \delta \sin \theta = 0. \] (3)

**Step 3. Achievable Periodic Solution of Reduced Dynamics.**
The system (3) describes the motions of a one-degree-of-freedom pendulum. Its phase portrait around the upward equilibrium and the corresponding solutions versus time are shown in the next two Figures.

**Step 4. Time-Periodic Auxiliary System.**
After feedback transformation one can linearize the system dynamics around the desired trajectory to get the time periodic LTV-system

\[ \frac{d}{dt} \begin{bmatrix} I \\ y \end{bmatrix} = \begin{bmatrix} \alpha_{11}(t) & 0 & \alpha_{13}(t) \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} I \\ y \end{bmatrix} + \begin{bmatrix} b_1(t) \\ 0 \\ 0 \end{bmatrix} v. \]

\[ [A, B] \] should be completely controllable over the period \( T \).

**Step 5. Control law.**
Solving the time-periodic Lur’e-Riccati equation

\[ \frac{d}{dt} B(t) + A(t)^T R(t) + R(t) A(t) + G = R(t) B(t) B(t)^T R(t)/T, \]

gives the control law

\[ v = -\Gamma^{-1} B(t)^T R(t) \left[ I \begin{bmatrix} y \end{bmatrix} \right]^T \]

with the generalized energy \( I \) given by

\[ I(\theta, \dot{\theta}, \theta_0(0), \dot{\theta}_0(0)) = \ddot{\theta}^2 + \Phi(\theta_0(0), \theta) \times \left[ \dot{\theta}_0(0)^2 - \int_0^\theta \Phi(s, \theta_0(0)) - \frac{2 \gamma \sin s}{\beta + \lambda K \cos s} ds \right]. \]

\[ \Phi(a, b) = \exp \left\{ \frac{b}{a} \beta K^2 \sin \tau \cos \tau \right\} \]

Experimental results

We performed experiments to test the proposed control design procedure, shaping stable oscillations of the pendulum around both the upright and downward equilibria [1]. Furthermore, the same design procedure is used for performing swing-up of the pendulum.

![Figure 2. (a) Phase-portrait trajectory of the pendulum oscillations around the upward equilibrium; (b) desired phase portrait.](image)

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