Robust Feedback Linearization using Extended High-Gain Observers

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Outline

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2. Extended High-Gain Observer
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Control Design via Feedback Linearization

Suppose after a certain change of coordinates we have:

\[
\dot{\xi} = A \xi + B \left( b(\xi, z, w) + a(\xi, z, w) u \right),
\]

\[
\dot{z} = \psi(\xi, z, w),
\]

\[
y = C x,
\]

where the triple \((A, B, C)\) represents a chain of integrators, i.e.

\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & \cdots & 0 \\
0 & 0 & 1 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\
0 & 0 & \cdots & \cdots & 1 & 0 \\
0 & 0 & \cdots & \cdots & 0 & 1 \\
0 & 0 & \cdots & \cdots & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
0 \\
1
\end{bmatrix}, \quad C = [1, 0, \cdots, 0].
Main Assumptions:

- $y$ is the only measured output, available for feedback,
- $w$ is a bounded differentiable signal with bounded derivatives,
- the sign of the high-frequency gain is known: $a(\xi, z, w) \geq \delta > 0$,
- the initial conditions belong to a known compact set,
- minimum-phase: $\dot{z} = \psi(\xi, z, w)$ is bounded-input-bounded-state (BIBS).
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Goal: ensure boundedness of all states, asymptotic smallness of the system output, and acceptable transient behavior.

Possible desired specifications are on: settling time, convergence rate, restricted overshoot.
Feedback Linearization with linear design:

Ignoring the zero-dynamics, the ideal feedback control design for

\[ \dot{\xi} = A\xi + B \left( b(\xi, z, w) + a(\xi, z, w) u \right), \]

is

\[ u = \left( -b(\xi, z, w) - K\xi \right) / a(\xi, z, w), \]

where \( K \) is such that \( A - BK \) is Hurwitz and the target system

\[ \dot{\bar{\xi}} = (A - BK) \bar{\xi}, \]

satisfies the desired specifications.
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The controller cannot be implemented since \( \xi, z, w \) are not available and \( a(\cdot), b(\cdot) \) are not known. Can we still achieve trajectories of the closed-loop system close to the ones of the target system?
Suppose, \( \hat{\xi} \) an estimate for \( \xi = [y', \ldots, y^{(n-1)}] \), provided by a high-gain observer, and \( \hat{a}(\hat{\xi}), \hat{b}(\hat{\xi}) \) are nominal models for \( a(\cdot), b(\cdot) \).

If we apply the corresponding approximation of the feedback linearization-based controller

\[
u = \left(-\hat{b}(\hat{\xi}) - K\hat{\xi}\right)/\hat{a}(\hat{\xi}),
\]

to our system we will obtain

\[
\dot{\xi} = (A - BK)\xi - B \Delta(\cdot), \quad \text{where} \quad \Delta(\cdot) \quad \text{is a certain discrepancy term}.
\]
Suppose, $\hat{\xi}$ an estimate for $\xi = [y', \ldots, y^{(n-1)}]$, provided by a high-gain observer, and $\hat{a}(\hat{\xi}), \hat{b}(\hat{\xi})$ are nominal models for $a(\cdot), b(\cdot)$.

If we apply the corresponding approximation of the feedback linearization-based controller

$$u = \left( -\hat{b}(\hat{\xi}) - K\hat{\xi} \right) / \hat{a}(\hat{\xi}),$$

to our system we will obtain $\dot{\xi} = (A - BK)\xi - B\Delta(\cdot)$, where $\Delta(\cdot)$ is a certain discrepancy term.

The difference from the target system is only in the last equation

$$y^{(n)} = -k_1 y' - \ldots - k_n y^{(n-1)} - \Delta(\cdot).$$

Can we additionally estimate $y^{(n)}$ and use it to compensate for $\Delta(\cdot)$?
Extended High-Gain Observer

For the main subsystem \( \dot{\xi} = A \xi + B \left( b(\xi, z, w) + a(\xi, z, w) u \right) \) we design the extended high-gain observer

\[
\begin{align*}
\dot{\hat{\xi}} &= A \hat{\xi} + B \left[ \hat{\sigma} + \hat{b}(\hat{\xi}) + \hat{a}(\hat{\xi}) u \right] + H(\varepsilon) \left( y - C \hat{\xi} \right), \\
\dot{\hat{\sigma}} &= \left( \alpha_{n+1}/\varepsilon^{n+1} \right) \left( y - C \hat{\xi} \right),
\end{align*}
\]

where

\[
H(\varepsilon) = \begin{bmatrix} \alpha_1/\varepsilon, \ldots, \alpha_n/\varepsilon^n \end{bmatrix}^T,
\]

with \( \alpha_1, \ldots, \alpha_n, \alpha_{n+1} \) being chosen such that the polynomial

\[
s^{n+1} + \alpha_1 s^n + \cdots + \alpha_{n+1}
\]

is Hurwitz and \( \varepsilon > 0 \) being a sufficiently small tuning parameter.
Assuming that the HGO achieves closeness of $\xi$ and $\hat{\xi}$, let us design a control law which ensures closeness of the main subsystem of the observer with $y - C\hat{\xi} \approx 0$, i.e. $\dot{\hat{\xi}} = A\hat{\xi} + B\left[\hat{\sigma} + \hat{b}(\hat{\xi}) + \hat{a}(\hat{\xi})u\right]$, to the target system.
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\end{align*}
\]
to the target system.

We want to have
\[
\bar{u} = \left( -\hat{\sigma} - \hat{b}(\hat{\xi}) - K \hat{\xi} \right) / \hat{a}(\hat{\xi}),
\]
but take
\[
u = \text{Sat}(\bar{u}) = \text{Sat} \left( \left[ -\hat{\sigma} - \hat{b}(\hat{\xi}) - K \hat{\xi} \right] / \hat{a}(\hat{\xi}) \right),
\]
where \( \text{Sat}(\cdot) \) is a continuously differentiable bounded function such that \( \text{Sat}(s) = s \) in the domain of interest, to protect the system from the destabilizing effect of peaking.
Theoretical Contribution

**Theorem 1.** Suppose all the functions above are continuously differentiable, $|\text{Sat}'(s)| \leq 1$ everywhere, and

$$|1 - a(\xi, z, w)/\hat{a}(\xi)| < 1/(2\alpha_{n+1}\|P\hat{B}\|),$$

where $P$ is a symmetric positive definite solution of $P \Lambda + \Lambda^T P = -I$ and $\Lambda$ is defined by the observer coefficients.

Then, there exists $\bar{\varepsilon} > 0$ such that for $\varepsilon \in (0, \bar{\varepsilon})$ every trajectory of the closed-loop system, initiated inside the given compact set, is bounded.

Moreover, uniformly for $t \geq 0$:

$$\|\xi(t) - \bar{\xi}(t)\| \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

where $\bar{\xi}(t)$ is the solution of the target system.
Key points of the proof:

Assuming that all functions are continuously differentiable,

\[ |\text{Sat}'(s)| \leq 1 \]

everywhere, and

\[ |1 - a(\xi, z, w)/\hat{a}(\xi)| \leq k_a < 1, \]

we can introduce the change of variables

\[ \eta_i = (\xi_i - \hat{\xi}_i)/\varepsilon^{n+1-i}, \quad \text{for} \quad 1 \leq i \leq n, \]

\[ \eta_{n+1} = b(\xi, z, w) - \hat{b}(\hat{\xi}) + \left( a(\xi, z, w) - \hat{a}(\hat{\xi}) \right) u - \hat{\sigma}, \]
to obtain the fast subsystem

\[ \varepsilon \dot{\eta} = \Lambda \eta + \bar{B} \left( \Delta_1(\cdot) + \varepsilon \Delta_2(\cdot) \right), \]

where

\[ \Delta_1(\cdot) = \left( 1 - a(\cdot)/\hat{a}(\cdot) \right) \text{Sat}'(\bar{u}) \alpha_{n+1} \eta_1. \]

Stability is not destroyed, provided \( k_\alpha < 1/(2\alpha_{n+1}\|P\bar{B}\|) \).
Example: DC motor with nonlinear friction

Let us consider the simplified model for a DC motor with an attached rigid arm

\[ J \ddot{y} = k_c u - F, \]

where

- \( y \) is the measured angle of the arm, which is to follow given bounded reference \( r(t) \) with two continuous bounded derivatives,
- \( J \) is the total moment of inertia of the rotor and the arm,
- \( k_c \) is the motor constant,
- \( u \) is the applied control signal, and
- \( F \) is an unknown friction torque, which can be either considered as a bounded differentiable disturbance or described by a LuGre model, which satisfies our BIBS assumption.
If we ignore \( F \neq 0 \), we take

\[
    u = \frac{\hat{J}}{\hat{k}_c} \text{Sat} \left( \ddot{e} - k_p (\hat{x}_1 - r) - k_d (\hat{x}_2 - \dot{r}) \right),
\]

where \( r(t) \) is the desired trajectory, \( k_p \) and \( k_d \) define the target system for \( e = x_1 - r \):

\[
    \ddot{e} + k_d \dot{e} + k_p e = 0
\]

and the HGO dynamics are given by

\[
    \dot{\hat{x}}_1 = \dot{\hat{x}}_2 + 2(y - \hat{x}_1)/\varepsilon,
\]

\[
    \dot{\hat{x}}_2 = (k_c/\hat{J}) u + (y - \hat{x}_1)/\varepsilon^2.
\]
Following the approach of this paper, we design the extended HGO:

\[
\begin{align*}
\dot{\xi}_1 &= \dot{\xi}_2 + 3(y - \dot{\xi}_1)/\varepsilon, \\
\dot{\xi}_2 &= \left(\frac{k_c}{\dot{J}}\right) u + \dot{\sigma} + 3(y - \dot{\xi}_1)/\varepsilon^2, \\
\dot{\sigma} &= 0 + (y - \dot{\xi}_1)/\varepsilon^3,
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and the output feedback control law

\[
u = \frac{\hat{J}}{\hat{k}_c} \text{Sat}\left( -\hat{\sigma} + \ddot{r} - k_p (\hat{\xi}_1 - r) - k_d (\hat{\xi}_2 - \dot{r}) \right).
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Note: this is a typical way of introducing an integral action via internal model principle to deal with unknown constant load.
Simulations and Experimental Results

Figure 1: Simulation results. (a) Parametric uncertainty: 19% mismatch. (b) Structural uncertainty: present LuGre friction. (c) Under the proposed controller.
Figure 2: Experimental results. Performance under a feedback-linearization controller with neglected friction.
Figure 3: Experimental results. Performance under the proposed controller.
Conclusion

- We have considered stabilization problem for feedback linearizable minimum-phase uncertain nonlinear control systems.
- We have designed an extended high-gain observer to estimate derivatives of all the states forming the chain of integrators in the global normal form.
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- We have designed an extended high-gain observer to estimate derivatives of all the states forming the chain of integrators in the global normal form.
- Using the output of the observer allows to recover performance achievable via exact feedback linearization followed by linear design strategy, which one would be able to implement directly if there were no uncertainty of the model and if all the states were available.
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- The proposed feedback control design has been verified both via numerical simulations and experimentally on an example of a DC motor with nonlinear friction.
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Thank you for your attention!