1 DC-motor dynamics

Dynamics of a DC motor with an attached rigid arm can be described by the simplified model
\[ J \ddot{\varphi} = k_e u - F, \]
where \( J \) is the total moment of inertia of the rotor and the arm, \( k_e \) is the motor constant, \( u \) is the applied control signal, and \( F \) is an unknown friction torque, modeled by the dynamic LuGre model
\[ F = z + \sigma_i (\varepsilon_0 z) + \sigma_2 \dot{\varphi}, \quad \varepsilon_0 z = \dot{\varphi} - \frac{|\dot{\varphi}|}{g(\dot{\varphi})}, \]
where \( \varepsilon_0 = 1/\sigma_0 \) is the reciprocal of the averaged stiffness of the bristles \( \sigma_0 \) (modeling contact between the surfaces), \( \sigma_i \) is the damping coefficient of the bristles, \( \sigma_2 \) is the viscous friction coefficient, and the Striebeck curve is defined by
\[ g(v) = \begin{cases} F_{c_1} + (F_{c_2} - F_{c_1}) e^{-v/\nu}, & v > 0, \\ F_{c_2} + (F_{c_1} - F_{c_2}) e^{-v/\nu}, & v < 0, \\ (g(0^+) + g(0^-))/2, & v = 0, \end{cases} \]
with \( F_{c_1} \) and \( F_{c_2} \) being the Coulomb and static friction values, and \( \nu \) being the Striebeck velocity. It is assumed that \(|z(0)| \leq \max(F_{c_1}, F_{c_2})\).

2 Control design for tracking with friction-free model

Let us take
\[ u = J \dot{\varphi} - k_e (\dot{x}_1 - 1) - k_d (\dot{x}_2 - v) / k_e, \]
where \( J, k_e, k_d, \) and \( \dot{\varphi} \) are a saturation function, \( r(t) \) is the desired trajectory, \( k_p = \omega_0^2 \), \( k_d = 2 \rho_0 \), \( \rho > 0.25 \), \( \omega_0 > 0 \), \( x_1 \) and \( x_2 \) are the states of the linear high-gain observer
\[ \dot{x}_1 = \dot{x}_2 + 2(\varphi - x_1)/\epsilon, \quad \dot{x}_2 = (\varphi - x_1)/\epsilon^2, \]
(4) If the initial conditions belong to a known compact set and the level of saturation is chosen appropriately, the trajectories of the closed-loop system (1), (3), (4) with \( F = 0 \) are such that
\[ \dot{e} + 2 \rho_0 \dot{e} + \omega_0^2 \epsilon \approx 0, \]
where \( e(t) = \phi(t) - r(t) \), up to small \( O(\epsilon) \) terms.

To check that neglecting friction could indeed be a problem, we test it on our experimental setup. We take: \( k_e = 2.5, J = 0.095, \varepsilon = 0.01, \)
\( r(t) = \alpha r \sin(\omega t), \alpha = 1, \omega = 2, \omega_0 = 1, \) and \( \eta = 0.7. \) The results are given on Fig. 1.

3 Friction compensator design

Let us take
\[ \ddot{\varphi} = \ddot{\varphi}, \quad \ddot{z} = \ddot{z} + K(\ddot{\varphi}), \]
where the friction parameters are assumed to be identified perfectly (for our setup: \( \sigma_2 = 0.004, F_{c_1} = 0.023 k_e, F_{c_2} = 0.021 k_e, F_{c_1} = 0.058 k_e, F_{c_2} = 0.052 k_e \)) and \( K(\cdot) \) is to be defined.

Consider the Lyapunov function candidate
\[ V(e_1, e_2, z - \dot{z}) = V(e_1, e_2) + \frac{\rho_0}{2} (z - \dot{z})^2 \]
where \( \rho > 0 \) and \( V(e, \dot{e}) \) is the strict Lyapunov function for (5):
\[ V(e_1, e_2) = \psi_0 (1 + \eta) e_1^2 + e_2^2 / \omega_0 \]
where \( e_1 = \varphi - r(t), e_2 = \dot{\varphi} - \dot{r}(t). \)
To ensure that the derivative of \( W \) along the trajectories of the closed-loop system (1), (6), and
\[ u = J (\ddot{x} - k_p e_1 - k_d e_2 + F) / k_e, \]
is negative, we can take
\[ K(e_1, e_2, v) = - (1 + \sigma_1 |v| g(v)) \frac{e_1 + 2 e_2 / \omega_0}{J \rho}. \]
For output feedback control we take instead of (6):
\[ \ddot{\varphi} = \ddot{\varphi}, \quad \ddot{z} = \ddot{z} + K(\ddot{\varphi}), \quad \ddot{z} = \ddot{z} + \dot{K}(\ddot{\varphi}) 
\]
where
\[ \ddot{K}(\cdot) = \left( \frac{1 + |\ddot{\varphi}|}{g(\ddot{\varphi})} \frac{e_1 + 2 e_2 / \omega_0}{J \rho}, \right), \]
where \( \rho > 0 \), combined with the linear high-gain observer (4) and
\[ u = J (\ddot{x} - k_p e_1 - k_d e_2 + F) / k_e. \]

4 Implementation issues

In order to implement (6) with any choice of \( K(\cdot) \) in real time with a sampled time \( h > 0 \), we need to obtain an appropriate sampled discrete version.
For the purpose of qualitative analysis we rewrite the differential part of (6) as
\[ \ddot{z} = -a(t) \ddot{z} + b(t), \]
where the functions
\[ a(t) = \left( |\ddot{\varphi}(t)| g(\ddot{\varphi}(t)) \right) / e_0 g(\ddot{\varphi}(t)) = 1, \]
and
\[ b(t) \approx h(k h) + K(e_1(k h), e_2(k h), \dot{\varphi}(k h)) = b_k \]
are assumed to be approximately constants during the \( k \)-th sampling period, i.e. for \( k h \leq t < k h + h \).
If one leaves discretization to the dSpace board (or to something similar), a fixed-step-based approximation would be used. To be specific, let us assume that the Euler method is used, so that
\[ \ddot{z}_{k+1} = (1 - a_k h) \ddot{z}_k - h b_k. \]
Alternatively, assuming that \( a(t) \equiv a_k \) and \( b(t) \equiv b_k \), we can integrate (11) exactly and obtain
\[ \ddot{z}_{k+1} = e^{-a_k h} \ddot{z}_k + e^{-a_k h} b_k - \frac{1}{a_k} b_k \]
where the singularity at \( a_k = 0 \) is removed.

Proposition 2 Suppose the sequence \( \{b_k\} \) is bounded. During any period of time when the speed exceeds the critical value
\[ v = \epsilon_0 / \max(F_{c_1}, F_{c_2}) \]
amost every solution of the discrete observer (12) oscillates and grows in magnitude. Moreover, if the speed is always greater than \( v \), all but one solutions of (12) escape to infinity while all the solutions of (2) and (6) with bounded \( K(\cdot) \) are globally bounded.

The difference equation (13) is stable and satisfies bounded-input bounded-state property, with the sequence \( \{b_k\} \) considered as an input, for arbitrary values of \( h > 0 \) and \( \alpha_k > 0 \).

Note that for our values of the parameters we need: \( |\ddot{z}| > v \approx 0.033 / h. \) Therefore, if we use the sampling time \( h = 0.001 \) seconds, we cannot allow the approximation (12) with speeds (output of the observer) greater than 1.3 radians per second.

5 Experimental validation

On Fig. 2 we show the experimental results obtained using (8), (9) with \( \rho = 10^6 \), discretized according to (13).

Figure 1: Experimental results. Performance under a feedback-linearization controller with neglected friction (controller is turned on at \( t \approx 1.9 \)).

Figure 2: Experimental results. Performance under the feedback-linearization controller with the passivity-based friction observer.