Transverse Linearization for Controlled Mechanical Systems with Several Passive Degrees of Freedom (Application to Orbital Stabilization)

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Problem Formulation \textit{(Euler-Lagrange systems)}

Key idea: \textit{virtual holonomic constraints}

Main result: \textit{transverse linearization and orbital stabilization}

Examples: \textit{spherical pendulum & synchronization of oscillations}

Outline

1. Problem Formulation \textit{(Euler-Lagrange systems)}

2. Key idea: \textit{virtual holonomic constraints}

3. Main result: \textit{transverse linearization and orbital stabilization}

4. Examples: \textit{spherical pendulum & synchronization of oscillations}

5. Summary
**Problem Formulation** *(Euler-Lagrange systems)*

**Key idea:** virtual holonomic constraints

**Main result:** transverse linearization and orbital stabilization

**Examples:** spherical pendulum & synchronization of oscillations

**Summary**
We consider the class of Euler-Lagrange systems

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q} = B(q) u
\]

\[
\begin{bmatrix}
M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + G(q) = B(q) u
\end{bmatrix}
\]

where \( q \in \mathbb{R}^n \) and \( \dot{q} \in \mathbb{R}^n \) are vectors of generalized coordinates and velocities, \( u \in \mathbb{R}^m \) is a vector of control inputs,

\[
\mathcal{L}(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q} - V(q)
\]

is the Lagrangian, \( M(q) \) is a positive-definite matrix of inertia, \( V(q) \) is a potential energy, and \( B(q) \) is a matrix of full-rank.

Challenges:

- underactuated, i.e. \( 0 \leq \text{dim } u = m < n = \text{dim } q \),
- not feedback linearizable and non minimum phase.

It is not clear what kinds of trajectories are possible and which can be stabilized.
Underactuated Mechanical Systems

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Goals of Analysis and/or Control Design

1. **Motion generation:**
   Design a control transformation
   \[ u = v + U(q, \dot{q}) \]
   such that there exist a nontrivial **periodic solution**
   \[ q = q_\star(t) = q_\star(t + T), \quad \forall t \geq 0, \quad (T > 0) \]
   of the closed-loop system with \( v \equiv 0 \), satisfying certain specifications (such as amplitude and period).

2. **Motion stabilization:**
   Design an **exponentially orbitally stabilizing** controller
   \[ v = f(q, \dot{q}), \quad f(q_\star(t), \dot{q}_\star) \equiv 0 \]
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Key idea: virtual holonomic constraints

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Examples: spherical pendulum & synchronization of oscillations

Summary
A periodic motion $q = q_\star(t) = [q_{1\star}(t), \ldots, q_{n\star}(t)]^T$ can be represented in a time-independent form

$$q_{1\star}(t) = \phi_1(\theta_{\star}(t)), \quad \ldots, \quad q_{n\star}(t) = \phi_n(\theta_{\star}(t))$$

where $\theta_{\star}$ is a parameterizing variable, describing a desired behavior of one of the generalized coordinates.

The induced family of functions

$$\{\phi_1(\cdot), \ldots, \phi_n(\cdot)\}$$

is called virtual holonomic constraints.

Let us consider a restriction of the dynamics consistent with these constraints, i.e. with

$$q_1 \equiv \phi_1(\theta), \quad \ldots, \quad q_n \equiv \phi_n(\theta)$$

where $\theta$ is treated as a new generalized coordinate.
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Computing the restricted dynamics

The dynamics consistent with

\[ q_1 \equiv \phi_1(\theta), \ldots, q_n \equiv \phi_n(\theta) \]

describes possible evolutions of \( \theta \) and can be computed:

**Lemma (Constrained dynamics)**

- **Independently on the generating control law, \( \theta \) is simultaneously a solution of \((n - m)\) differential equations**

\[
\alpha_i(\theta) \ddot{\theta} + \beta_i(\theta) \dot{\theta}^2 + \gamma_i(\theta) = 0, \quad i = 1, \ldots, n - m
\]

some of which can be of lower order or even trivial.

- **One of the solutions is** \( \theta = \theta_*(t) \).

- **For each \( i \) such that \( \alpha_i(\theta) \neq 0 \), there is a conserved quantity**

\[
I^{(i)}(\theta(0), \dot{\theta}(0), \theta(t), \dot{\theta}(t)).
\]
The projected dynamics is

\[ \alpha_i(\theta) \ddot{\theta} + \beta_i(\theta) \dot{\theta}^2 + \gamma_i(\theta) = 0, \quad i = 1, \ldots, n - m \]

or in an algebraic form

\[ I^{(i)} \left( \theta(0), \dot{\theta}(0), \theta(t), \dot{\theta}(t) \right) = 0, \quad i = 1, \ldots, n - m \]

provided \( \alpha_i(\theta) \neq 0 \).

1. All the functions \( \alpha_i(\cdot), \beta_i(\cdot), \gamma_i(\cdot), \) and \( I^{(i)}(\cdot) \) can be computed analytically from the expression for the Lagrangian \( \mathcal{L}(q, \dot{q}) \).

2. Since existence of a periodic solution for the restricted dynamics is equivalent to the existence of a periodic solution for the Euler-Lagrange equations consistent with the constraints, it can be used for motion planning, not only for stability analysis/stabilization to be discussed.
Problem Formulation *(Euler-Lagrange systems)*
Key idea: *virtual holonomic constraints*
Main result: *transverse linearization and orbital stabilization*
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Transverse dynamics to linearize

We want to find new variables in a vicinity of a *target periodic trajectory* such that the system’s states are decomposed into:

1. A scalar variable representing position *along the trajectory*.
2. The remaining \((2n - 1)\) variables representing the dynamics *transverse to the trajectory* and defining a **moving Poincaré section** \(S^{(i)}(t), \ t \in [0, T]\).

Figure: Each surface \(S^{(i)}(t)\) transversal to the flow (locally, around the intersection with the cycle) is called a Poincaré section.
Looking for good conserved quantities

In order to find transverse coordinates, one just needs $(2n - 1)$ independent quantities conserved along the periodic trajectory.

However, they should be chosen in such a way that it is possible to **compute linearization of the transverse dynamics** along the periodic trajectory **analytically**.

Suppose virtual holonomic constraints are given. They are conserved along the cycle!

So, let us introduce the following change of coordinates

\[
\begin{align*}
(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) & \leftrightarrow (y_1, \ldots, y_{n-1}, \dot{y}_1, \ldots, \dot{y}_{n-1}, \theta, \dot{\theta}) \\
q_1 = y_1 + \phi_1(\theta), & \quad \ldots, & \quad y_{n-1} = q_{n-1} + \phi_{n-1}(\theta).
\end{align*}
\]
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\end{array} \right) \leftrightarrow \left( \begin{array}{c}
y_1, \ldots, y_{n-1}, \dot{y}_1, \ldots, \dot{y}_{n-1}, \theta, \dot{\theta}
\end{array} \right)
\]

\[
y_1 = q_1 - \phi_1(\theta), \quad \ldots, \quad y_{n-1} = q_{n-1} - \phi_{n-1}(\theta).
\]
The Euler-Lagrange dynamics can be locally rewritten as

\[ \alpha_i(\theta) \ddot{\theta} + \beta_i(\theta) \dot{\theta}^2 + \gamma_i(\theta) = w, \]

\[ w = g_y^{(i)}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y}) \ y + g_{\dot{y}}^{(i)}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y}) \ \dot{y} + g_v^{(i)}(\theta, \dot{\theta}, y, \dot{y}) \ v \]

\[ \ddot{y} = F(\theta, \dot{\theta}, y, \dot{y}) + N(\theta, y) \ v. \]

where \( F(\theta_*, \dot{\theta}_*, 0, 0) \equiv 0 \) and \( i \) is such that \( \alpha_i(\theta) \neq 0 \).

For the rewritten dynamics the following quantities

\[ y_1, \ldots, y_{n-1}, \dot{y}_1, \ldots, \dot{y}_{n-1}, I^{(1)}(\cdot), \ldots, I^{(n-m)}(\cdot) \]

are conserved along the target trajectory as well as their combinations and derivatives.

We need to choose those independent \((2n-1)\) of them dynamics of which can be linearized analytically.
Theorem (Rewritten dynamics)

*The Euler-Lagrange dynamics can be locally rewritten as*

\[\alpha_i(\theta)\ddot{\theta} + \beta_i(\theta)\dot{\theta}^2 + \gamma_i(\theta) = w,\]

\[w = g^{(i)}_y(\theta,\dot{\theta},\ddot{\theta},y,\dot{y}) y + g^{(i)}_\dot{y}(\theta,\dot{\theta},\ddot{\theta},y,\dot{y}) \dot{y} + g^{(i)}_v(\theta,\dot{\theta},y,\dot{y}) v\]

\[\ddot{y} = F(\theta, \dot{\theta}, y, \dot{y}) + N(\theta, y) v.\]

*where* \[F(\theta_*, \dot{\theta}_*, 0, 0) \equiv 0\] *and* \(i\) *is such that* \[\alpha_i(\theta) \neq 0.\]

For the rewritten dynamics the following quantities

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*are conserved along the target trajectory as well as their combinations and derivatives.*

*We need to choose those independent* \((2n - 1)\) *of them dynamics of which can be linearized analytically.*
Dynamics in deviations from the constrained manifold

**Theorem (Rewritten dynamics)**

The Euler-Lagrange dynamics can be locally rewritten as

\[
\alpha_i(\theta) \ddot{\theta} + \beta_i(\theta) \dot{\theta}^2 + \gamma_i(\theta) = w,
\]

\[
w = g_y^{(i)}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y}) \cdot y + g_y^{(i)}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y}) \cdot \dot{y} + g_v^{(i)}(\theta, \dot{\theta}, y, \dot{y}) \cdot \dot{y}
\]

\[
\ddot{y} = F(\theta, \dot{\theta}, y, \dot{y}) + N(\theta, y) \cdot \nu.
\]

where \( F(\theta_\star, \dot{\theta}_\star, 0, 0) \equiv 0 \) and \( i \) is such that \( \alpha_i(\theta) \neq 0 \).

For the rewritten dynamics the following quantities

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are conserved along the target trajectory as well as their combinations and derivatives.

We need to choose those independent \((2n - 1)\) of them dynamics of which can be linearized analytically.
A choice of transverse coordinates

The dynamics for $y$ and $\dot{y}$

$$\ddot{y} = F(\theta, \dot{\theta}, y, \dot{y}) + N(\theta, y) \nu$$

is easy to linearize since

$$\frac{\partial F}{\partial I^{(i)}} = \left( \frac{\partial F}{\partial \dot{\theta} \dot{\theta}} \dot{\theta} - \frac{\partial F}{\partial \theta} \ddot{\theta} \right) \bigg/ (2\dot{\theta}^2 + 2\ddot{\theta}^2)$$

For the rest, the following identity is useful

$$\frac{d}{dt} I^{(i)} \left( \theta_*(0), \dot{\theta}_*(0), \dot{\theta}, \dot{\theta} \right) = \frac{2 \dot{\theta}}{\alpha(\theta)} \left( w - \beta(\theta) I^{(i)}(\cdot) \right)$$

Introduce another (conceptual) change of coordinates

$$\left( y, \dot{y}, \theta, \dot{\theta} \right) \leftrightarrow \left( y, \dot{y}, I^{(i)}, \psi^{(i)} \right)$$

The target periodic trajectory transforms as follows

$$\begin{align*}
q &= q_*(t) \\
\dot{q} &= \dot{q}_*(t)
\end{align*} \leftrightarrow \begin{align*}
\theta &= \theta_*(t) \\
\dot{\theta} &= \dot{\theta}_*(t) \\
y &= 0 \\
\dot{y} &= 0
\end{align*} \leftrightarrow \begin{align*}
\psi^{(i)} &= \psi_*(t) \\
I^{(i)} &= 0 \\
y &= 0 \\
\dot{y} &= 0
\end{align*}$$
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The dynamics for $y$ and $\dot{y}$

$$\ddot{y} = F(\theta, \dot{\theta}, y, \dot{y}) + N(\theta, y) v$$

is easy to linearize since

$$\frac{\partial F}{\partial I^{(i)}} = \left( \frac{\partial F}{\partial \dot{\theta}} \dot{\theta}_{\star} - \frac{\partial F}{\partial \theta} \ddot{\theta}_{\star} \right) / (2\dot{\theta}_{\star}^2 + 2\ddot{\theta}_{\star}^2)$$

For the rest, the following identity is useful

$$\frac{d}{dt} I^{(i)}(\theta_{\star}(0), \dot{\theta}_{\star}(0), \dot{\theta}, \dot{\theta}) = \frac{2 \dot{\theta}}{\alpha(\theta)} \left( w - \beta(\theta) I^{(i)}(\cdot) \right)$$

Introduce another (conceptual) change of coordinates

$$(y, \dot{y}, \theta, \dot{\theta}) \leftrightarrow (y, \dot{y}, I^{(i)}, \psi^{(i)})$$

The target periodic trajectory transforms as follows

$$\begin{align*}
\{ q = q_{\star}(t) \} & \leftrightarrow \{ \theta = \theta_{\star}(t) \} \\
\{ \dot{q} = \dot{q}_{\star}(t) \} & \leftrightarrow \{ \dot{\theta} = \dot{\theta}_{\star}(t) \} \\
\{ y = 0 \} & \leftrightarrow \{ \psi^{(i)} = \psi_{\star}(t) \} \\
\{ \dot{y} = 0 \} & \leftrightarrow \{ I^{(i)} = 0 \} \\
\{ \dot{\psi} = 0 \} & \leftrightarrow \{ \dot{y} = 0 \} \\
\{ \ddot{\psi} = 0 \} & \leftrightarrow \{ \ddot{y} = 0 \}
\end{align*}$$
A choice of transverse coordinates

The dynamics for $\mathbf{y}$ and $\dot{\mathbf{y}}$

$$\ddot{\mathbf{y}} = \mathbf{F}(\theta, \dot{\theta}, \mathbf{y}, \dot{\mathbf{y}}) + \mathbf{N}(\theta, \mathbf{y}) \mathbf{v}$$

is easy to linearize since $\frac{\partial \mathbf{F}}{\partial \mathbf{l}^{(i)}} = \left( \frac{\partial \mathbf{F}}{\partial \dot{\theta}} \dot{\theta}_* - \frac{\partial \mathbf{F}}{\partial \theta} \ddot{\theta}_* \right) / (2\dot{\theta}_*^2 + 2\ddot{\theta}_*)$

For the rest, the following identity is useful

$$\frac{d}{dt} \mathbf{l}^{(i)} \left( \theta_*(0), \dot{\theta}_*(0), \dot{\theta}, \ddot{\theta} \right) = \frac{2 \dot{\theta}}{\alpha(\theta)} \left( \mathbf{w} - \beta(\theta) \mathbf{l}^{(i)}(\cdot) \right)$$

Introduce another (conceptual) change of coordinates

$$\left( \mathbf{y}, \dot{\mathbf{y}}, \theta, \dot{\theta} \right) \leftrightarrow \left( \mathbf{y}, \dot{\mathbf{y}}, \mathbf{l}^{(i)}, \psi^{(i)} \right)$$

The target periodic trajectory transforms as follows

$$\begin{cases}
\mathbf{q} = \mathbf{q}_*(t) \\ \dot{\mathbf{q}} = \dot{\mathbf{q}}_*(t)
\end{cases} \leftrightarrow \begin{cases}
\theta = \theta_*(t) \\
\dot{\theta} = \dot{\theta}_*(t)
\end{cases} \leftrightarrow \begin{cases}
\psi^{(i)} = \psi_*(t) \\
\mathbf{l}^{(i)} = 0 \\
\mathbf{y} = 0 \\
\dot{\mathbf{y}} = 0
\end{cases}$$
Linear deviations

A possible choice of transverse coordinates:

\[ \mathbf{x}_\perp = \left[ I^{(i)}, \mathbf{y}^T, \dot{\mathbf{y}}^T \right]^T \in \mathbb{R}^{2n-1} \]

The variable \( \psi^{(i)} \) defines location along the periodic trajectory and the moving Poincaré section:

\[ S^{(i)}(t) := \left\{ [\theta, \dot{\theta}, \mathbf{y}; \dot{\mathbf{y}}] \in \mathbb{R}^{2n} : \psi^{(i)}(\theta, \dot{\theta}) = \psi^{(i)}_{\ast}(t) \right\} \]

Let

\[
\begin{bmatrix}
I^{(i)}_0 \\
Y_{10} \\
Y_{20}
\end{bmatrix}
\in TS^{(i)}
\]

denotes linearization for

\[
\begin{bmatrix}
I^{(i)}_0 \\
\mathbf{y} \\
\dot{\mathbf{y}}
\end{bmatrix}
\in S^{(i)}
\]

\[ TS^{(i)}(t) := \left\{ [q^T, \dot{q}^T]^T : (q-q_{\ast}(t))^T \dot{q}_{\ast}(t) + (\dot{q}-\dot{q}_{\ast}(t))^T \ddot{q}_{\ast}(t) = 0 \right\} \]
Transverse linearization

Keeping the first-order terms in the Taylor multiseries expansions, the linearization of the dynamics for the transverse coordinates takes the form of the time-periodic linear system

$$\frac{d}{dt} X_\bullet(t) = A(t) X_\bullet(t) + B(t) V_\bullet$$

with the following structure:

$$\frac{d}{dt} l^{(i)} = a_{11}^{(i)}(t) l^{(i)} + a_{12}^{(i)}(t) Y_{1\bullet} + a_{13}^{(i)}(t) Y_{2\bullet} + b_{1}^{(i)}(t) V_\bullet$$

$$\frac{d}{dt} \begin{bmatrix} Y_{1\bullet} \\ Y_{2\bullet} \end{bmatrix} = \begin{bmatrix} 0_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} & 1_{(n-1)\times(n-1)} \\ A_{21}(t) & A_{22}(t) & A_{23}(t) \end{bmatrix} \begin{bmatrix} l^{(i)} \\ Y_{1\bullet} \\ Y_{2\bullet} \end{bmatrix} + \begin{bmatrix} 0_{(n-1)\times 1} \\ B_{2}(t) \end{bmatrix} V_\bullet$$

How to use it for orbital stabilization?
Theorem (Exponential orbital stabilization)

The following two statements are equivalent.

1. There is a $C^1$-smooth $K(t)$ such that

$$V_\bullet = K(t) \left[ I_{\bullet}^{(i)}, \ Y_{1\bullet}, \ Y_{2\bullet} \right]^T, \ K(t) = K(t + T)$$

stabilizes the origin of the transverse linearization.

2. There exists a $C^1$-smooth $v = f(q, \dot{q})$ that ensures exponentially orbitally stability.

Furthermore, two possible choices for $v$ are

$$v(t) = K(\tau) \left[ I^{(i)}, \ y^T, \ \dot{y}^T \right]^T$$

$$\tau = \left\{ s : \left[ q^T(t), \ \dot{q}^T(t) \right]^T \in \begin{bmatrix} S^{(i)}(s) \\ \text{or} \\ TS^{(i)}(s) \end{bmatrix} \right\} \cap O_\varepsilon(q_\star)$$

where $O_\varepsilon(q_\star)$ is a small tube around the trajectory.
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Stable oscillations of a spherical pendulum on a puck

Achieved goal: exponentially orbitally stable oscillations of the pendulum around the upright equilibrium.

Figure: A spherical pendulum on a puck. The coordinates $x_1$ and $x_2$ represent the position of the puck in the horizontal plane; the angles $\varepsilon_1$ and $\varepsilon_2$ give the orientation with respect to the inertia frame.

Dynamics can be described as follows

$$\frac{d}{dt} \left[ \frac{\partial L(\cdot)}{\partial \dot{\varepsilon}_{\{1,2\}}} \right] - \frac{\partial L(\cdot)}{\partial \varepsilon_{\{1,2\}}} = 0,$$

$$\frac{d}{dt} \left[ \frac{\partial L(\cdot)}{\partial \dot{x}_{\{1,2\}}} \right] - \frac{\partial L(\cdot)}{\partial x_{\{1,2\}}} = u_{\{1,2\}},$$
Synchronization of oscillations of pendula on carts

Goal: synchronous exponentially orbitally stable oscillations of the cart-pendulum systems around their upright equilibria.

Figure: Three identical cart-pendulum systems. The coordinates $x_1$, $x_2$, and $x_3$ represent positions of the carts along the horizontal; $\theta_1$, $\theta_2$, and $\theta_3$ give the angles of the pendula with respect to the vertical. Dynamics can be described as follows

$$2 \ddot{x}_i + \cos(\theta_i) \ddot{\theta}_i - \sin(\theta_i) \dot{\theta}_i^2 = u_i,$$

$$\cos(\theta_i) \ddot{x}_i + \ddot{\theta}_i - g \sin(\theta_i) = 0,$$  

$i = 1, \ldots, N$
Synchronization of oscillations: numerical simulations

Figure: Synchronization of oscillations for 3 cart-pendulum systems: An evolution of the angles – the $\theta_1(t)$, $\theta_2(t)$, $\theta_3(t)$-variables – along the solution of the closed-loop system. They are synchronized in about one period $\approx 5$ sec and have reached after transition the target amplitude of oscillations of $0.2\text{[rad]}$. 
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- Class of models: *underactuated Euler-Lagrange systems*.

- Goal: *to create orbitally exponentially stable periodic motions*.

- Approach: *finding linearizable transverse coordinates and stabilizing their linearized dynamics*.

- Key: *using deviations from geometric relations along the target periodic motion and a conserved quantity of the restricted dynamics*.

- Possibility: *creating synchronous oscillations*. 

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