Given: \( y(t) = G_\theta(p) u(t) \), \( y_m(t) = G_m(p) u_c(t) \),

Find: \( u(t) = -\frac{S_\dot{\theta}(p)}{R_\dot{\theta}(p)} y(t) + \frac{T_\dot{\theta}(p)}{R_\dot{\theta}(p)} u_c(t) \), \( \frac{d}{dt} \dot{\theta} = \ldots \)
MIT rule (Example 5.1)

Consider a stable single input single output (SISO) system

\[ y(t) = k \cdot G(p)(u(t)) \]

where

- \( y(t) \) is the system output,
- \( G(s) \) is a known stable transfer function,
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The problem is find a controller \( u(t) = \frac{T(p)}{R(p)} u_c(t) \) to follow

\[ y_m(t) = G_m(p) \left( u_c(t) \right) = k_0 \cdot G(p) \left( u_c(t) \right), \]

where \( k_0 \) is a given constant gain.
MIT rule (Example 5.1)

If $k$ were known we can solve the problem

$$y(t) = k \cdot G(p) \left( u(t) \right) \quad \rightarrow \quad y_m(t) = k_0 \cdot G(p) \left( u_c(t) \right)$$

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This is, indeed, works, because if \( \theta \) is chosen as

\[
\theta = \frac{k_0}{k}
\]

then

\[
y(t) = k \cdot G(p) \left( \theta \cdot u_c(t) \right) = k \cdot G(p) \left( \frac{k_0}{k} \cdot u_c(t) \right) = y_m(t)
\]
MIT rule (Example 5.1)

Question: How to update (adapt) the value of $\theta$?
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Let us consider the error between the real and simulated outputs

$$e(t, \theta) = y(t) - y_m(t) = k \cdot G(p) \left( \theta(t) \cdot u_c(t) \right) - k_0 \cdot G(p) \left( u_c(t) \right)$$
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Let us update $\theta(t)$ so that $e(t, \theta)$ is getting smaller.
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Let us update $\theta(t)$ so that $e(t, \theta)$ is getting smaller.

Consider the loss function that measures the size of $e(t, \theta)$

$$\mathcal{J}(t, \theta) = |e(t, \theta)|^2$$

Its time derivative is given by (chain rule)

$$\frac{d}{dt} \mathcal{J} = \left[ \frac{\partial}{\partial t} \mathcal{J} \right] + \left[ \frac{\partial}{\partial \theta} \mathcal{J} \right] \cdot \frac{d}{dt} \theta$$
MIT rule (Example 5.1)

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Consider the function that measures the size of $e(t, \theta)$

$$\mathcal{J}(t, \theta) = |e(t, \theta)|^2$$

Its time derivative should be made negative:

$$\frac{d}{dt} \mathcal{J} = \cdots + \left[ \frac{\partial}{\partial \theta} \mathcal{J} \right] \cdot \frac{d}{dt} \theta \quad \Rightarrow \quad \frac{d}{dt} \theta = -\gamma \cdot \left[ \frac{\partial}{\partial \theta} \mathcal{J} \right]$$
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Its time derivative should be made negative:

$$\frac{d}{dt} \mathcal{J} = \cdots + \left[ 2 \, e \frac{\partial}{\partial \theta} e \right] \cdot \frac{d}{dt} \theta \quad \Rightarrow \quad \frac{d}{dt} \theta = -\gamma \cdot \left[ 2 \, e \frac{\partial}{\partial \theta} e \right]$$
MIT rule (Example 5.1)

Computing the partial derivative of $e$ wrt $\theta$ we have

$$\frac{\partial e}{\partial \theta} = \frac{\partial}{\partial \theta} \left[ k \cdot G(p) \left( \theta(t) \cdot u_c(t) \right) \right] = k \cdot G(p) \left( u_c(t) \right) = \frac{k}{k_0} \cdot k_0 \cdot G(p) \left( u_c(t) \right) = \frac{k}{k_0} y_m(t)$$
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$$\frac{\partial}{\partial \theta} e = \frac{\partial}{\partial \theta} \left[ k \cdot G(p) \left( \theta(t) \cdot u_c(t) \right) \right] = k \cdot G(p) \left( u_c(t) \right)$$

$$= \frac{k}{k_0} \cdot G(p) \left( u_c(t) \right) = \frac{k}{k_0} y_m(t)$$

Then the update law for $\theta$ becomes

$$\frac{d}{dt} \theta = -\gamma \cdot \left[ 2 e \frac{\partial}{\partial \theta} e \right] = -\gamma_n \cdot y_m(t) \cdot e(t, \theta)$$

where $\gamma_n > 0$ is arbitrary since $\gamma_n = \gamma \frac{k}{k_0}$ with arbitrary $\gamma > 0$.  

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where $\gamma_n > 0$ is arbitrary since $\gamma_n = \gamma \frac{k}{k_0}$ with arbitrary $\gamma > 0$.

Remark: $\mathcal{J}(\cdot) = |e(\cdot)| \Rightarrow \frac{d}{dt} \theta = -\gamma_n \cdot y_m(t) \cdot \text{sign}[e(t, \theta)]$. 
MIT rule (Example 5.1)

Suppose that

\[ G(s) = \frac{1}{s + 1} \]

and

\[ k = 1, \quad k_0 = 2 \]
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The update law for \( \theta \)

\[ \frac{d}{dt} \theta = -\gamma \cdot \left[ 2e \frac{\partial}{\partial \theta} e \right] = -\gamma_n \cdot y_m(t) \cdot e(t) \]

is simulated for various values of

\[ \gamma = 0.5, \quad 2.5, \quad 4.5 \]

with \( u_c(t) = \sin t \).
The behavior of $\theta$ for various values of $\gamma$. 
The behavior of $y(t)$ for various values of $\gamma$. 

MIT rule (Example 5.2)

Suppose that the system dynamics are

\[ \frac{d}{dt}y = -ay + bu, \quad y = \frac{b}{p + au} \]
MIT rule (Example 5.2)

Suppose that the system dynamics are

\[
\frac{d}{dt} y = -a \, y + b \, u, \quad y = \frac{b}{p + a} \, u
\]

While the desired dynamics for the closed loop system is

\[
\frac{d}{dt} y_m = -a_m \, y_m + b_m \, u_c, \quad y_m = \frac{b_m}{p + a_m} \, u_c
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\]

The proportional controller that solves the problem is given by
\[
u(t) = \frac{T(p)}{R(p)} u_c(t) - \frac{S(p)}{R(p)} y(t) = \theta_1 u_c(t) - \theta_2 y(t)
\]
MIT rule (Example 5.2)

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$$\frac{d}{dt} y = -a y + b u, \quad y = \frac{b}{p + a} u$$

While the desired dynamics for the closed loop system is

$$\frac{d}{dt} y_m = -a_m y_m + b_m u_c, \quad y_m = \frac{b_m}{p + a_m} u_c$$

The proportional controller that solves the problem is given by

$$u(t) = \theta_1 u_c(t) - \theta_2 y(t)$$

where the gains to ensure the desired system response are

$$\theta_1 = \theta_1^0 = \frac{b_m}{b}, \quad \theta_2 = \theta_2^0 = \frac{a_m - a}{b}$$
MIT rule (Example 5.2)

Introduce the error signal

\[ e(t) = y(t) - y_m(t) = \frac{b \theta_1}{p + a + b \theta_2} u_c(t) - \frac{b_m}{p + a_m} u_c \]
MIT rule (Example 5.2)

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\[ e(t) = y(t) - y_m(t) = \frac{b \theta_1}{p + a + b \theta_2} u_c(t) - \frac{b_m}{p + a} u_c \]

Computing partial derivatives of \( e(\cdot) \) w.r.t. \( \theta_1, \theta_2 \), we have

\[ \frac{\partial e}{\partial \theta_1} = \frac{b}{p + a + b \theta_2} u_c(t) \]

\[ \frac{\partial e}{\partial \theta_2} = -\frac{b \theta_1 \cdot b}{(p + a + b \theta_2)^2} u_c(t) = -\frac{b}{p + a + b \theta_2} y(t) \]
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Such formulas cannot be used for updating \( \theta_1 \) and \( \theta_2 \) values.

Indeed, constants \( a \) and \( b \) are not known!
MIT rule (Example 5.2)

Introduce the error signal

\[ e(t) = y(t) - y_m(t) = \frac{b \theta_1}{p + a + b \theta_2} u_c(t) - \frac{b_m}{p + a_m} u_c \]

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\]

\[
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\]

Not much can be done, we will assume that we can initialize \( \theta_2 \) around its nominal value

\[ \theta_2 \approx \theta_2^0 = \frac{a_m - a}{b} \]
MIT rule (Example 5.2)

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\[ e(t) = y(t) - y_m(t) = \frac{b \theta_1}{p + a + b \theta_2} u_c(t) - \frac{b_m}{p + a_m} u_c \]

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\]

\[
\frac{\partial e}{\partial \theta_2} = -\frac{b \theta_1 \cdot b}{(p + a + b \theta_2)^2} u_c(t) = \frac{-b}{p + a + b \theta_2} y(t)
\]

Not much can be done, we will assume that we can initialize \( \theta_2 \) around its nominal value

\[
\theta_2 \approx \theta_2^0 = \frac{a_m - a}{b} \quad \Rightarrow \quad (a + b \theta_2) \approx a_m
\]
This results in the following relations

\[
\frac{d}{dt} \theta_1 = -\gamma \cdot e(t) \cdot \frac{\partial e(t)}{\partial \theta_1} \approx -\gamma \cdot e(t) \cdot \left( \frac{b}{s + a_m} u_c(t) \right)
\]

\[
= -\gamma_n \cdot e(t) \cdot \left( \frac{a_m}{s + a_m} u_c(t) \right)
\]
MIT rule (Example 5.2)

This results in the following relations

\[
\frac{d}{dt}\theta_1 \approx -\gamma_n \cdot e(t) \cdot \left( \frac{a_m}{s + a_m} u_c(t) \right)
\]

\[
\frac{d}{dt}\theta_2 = -\gamma \cdot e(t) \cdot \frac{\partial e(t)}{\partial \theta_2} \approx -\gamma \cdot e(t) \cdot \left( \frac{-b}{s + a_m} y(t) \right)
\]

\[
\approx \gamma_n \cdot e(t) \cdot \left( \frac{a_m}{s + a_m} y(t) \right)
\]

where

\[
\gamma_n = \gamma \frac{b}{a_m}
\]

and should be positive!
MIT rule (Example 5.2)

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\[
\frac{d}{dt}\theta_1 \approx -\gamma_n \cdot e(t) \cdot \left( \frac{a_m}{s + a_m} u_c(t) \right)
\]

\[
\frac{d}{dt}\theta_2 = -\gamma \cdot e(t) \cdot \frac{\partial e(t)}{\partial \theta_2} \approx -\gamma \cdot e(t) \cdot \left( \frac{-b}{s + a_m} y(t) \right)
\]

\[
= \gamma_n \cdot e(t) \cdot \left( \frac{a_m}{s + a_m} y(t) \right)
\]

where

\[
\gamma_n = \gamma \frac{b}{a_m}
\]

and should be positive!

To implement this algorithm we need to know the sign of \( b \)!  

Figure 5.4 Block diagram of a model-reference controller for a first-order process.

Figure 5.5 Simulation of the system in Example 5.2 using an MRAS. The parameter values are $a = 1$, $b = 0.5$, $a_m = b_m = 2$, and $\gamma = 1$. 
Figure 5.6  Controller parameters $\theta_1$ and $\theta_2$ for the system in Example 5.2 when $\gamma = 0.2, 1$ and 5.

Figure 5.7  Relation between controller parameters $\theta_1$ and $\theta_2$ when the system in Example 5.2 is simulated for 500 time units. The dashed line shows the line $\theta_2 = \theta_1 - a/b$. The dot indicates the convergence point.
MIT rule (Example 5.3)

Consider the static system with unknown gain $k$

$$y(t) = k \cdot u(t), \quad G(s) \equiv 1$$

and the problem of amplifying $u_c(t)$ so that we match

$$y_m(t) = k_0 \cdot u_c(t)$$
MIT rule (Example 5.3)

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$$y(t) = k \cdot u(t), \quad G(s) \equiv 1$$

and the problem of amplifying $u_c(t)$ so that we match

$$y_m(t) = k_0 \cdot u_c(t)$$

With $u(t) = \theta u_c(t)$ introduce the error

$$e(t) = y(t) - y_m(t) = k \cdot (\theta u_c(t)) - k_0 \cdot u_c(t) = k (\theta - \theta^0) u_c(t)$$

with $\theta^0 = k_0 / k$. 
MIT rule (Example 5.3)

Consider the static system with unknown gain $k$

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$$e(t) = y(t) - y_m(t) = k \cdot (\theta \cdot u_c(t)) - k_0 \cdot u_c(t) = k (\theta - \theta^0) \cdot u_c(t)$$

with $\theta^0 = k_0 / k$.

$$\frac{d}{dt} \theta(t) = -\gamma \cdot e(t) \cdot \frac{\partial e(t)}{\partial \theta} = -\gamma \cdot k (\theta(t) - \theta^0) \cdot u_c(t) \cdot k \cdot u_c(t)$$
MIT rule (Example 5.3)

Consider the static system with unknown gain $k$

$$y(t) = k \cdot u(t), \quad G(s) \equiv 1$$

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$$e(t) = y(t) - y_m(t) = k \cdot (\theta u_c(t)) - k_0 \cdot u_c(t) = k (\theta - \theta^0) u_c(t)$$

with $\theta^0 = k_0 / k$.

$$\frac{d}{dt} \theta(t) = -\gamma \cdot k^2 \cdot (u_c(t))^2 \cdot (\theta(t) - \theta^0)$$
MIT rule (Example 5.3)

Consider the static system with unknown gain $k$

$$y(t) = k \cdot u(t), \quad G(s) \equiv 1$$

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$$e(t) = y(t) - y_m(t) = k \cdot (\theta \cdot u_c(t)) - k_0 \cdot u_c(t) = k (\theta - \theta^0) \cdot u_c(t)$$

with $\theta^0 = \frac{k_0}{k}$.

$$\frac{d}{dt} (\theta(t) - \theta^0) = -\gamma_n \cdot k \cdot (u_c(t))^2 \cdot (\theta(t) - \theta^0)$$
MIT rule (Example 5.3)

Consider the static system with unknown gain \( k \)

\[
    y(t) = k \cdot u(t), \quad G(s) \equiv 1
\]

and the problem of amplifying \( u_c(t) \) so that we match

\[
    y_m(t) = k_0 \cdot u_c(t)
\]

With \( u(t) = \theta \ u_c(t) \) introduce the error

\[
    e(t) = y(t) - y_m(t) = k \cdot (\theta \ u_c(t)) - k_0 \cdot u_c(t) = k \ (\theta - \theta^0) \ u_c(t)
\]

with \( \theta^0 = k_0 / k \).

\[
    (\theta(t) - \theta^0) = \exp \left\{ -\gamma_n \cdot k \cdot \int_0^t (u_c(t))^2 \, d\tau \right\} \cdot (\theta(0) - \theta^0)
\]
MIT rule (Example 5.3)

Consider the static system with unknown gain $k$

$$y(t) = k \cdot u(t), \quad G(s) \equiv 1$$

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with $\theta^0 = \frac{k_0}{k}$.

$$e(t) = k \cdot \exp \left\{ -\gamma_n \cdot k \cdot \int_0^t (u_c(t))^2 d\tau \right\} \cdot (\theta(0) - \theta^0) \cdot u_c(t)$$

$\theta(t) - \theta^0$
MIT rule (Example 5.3), cont’d

For the system and model given by

\[ y(t) = k \cdot u(t), \quad y_m(t) = k_0 \cdot u_c(t) \]

we define \( e(t) = y(t) - y_m(t) \) and take

\[ u(t) = \theta(t) \cdot u_c(t), \quad \frac{d}{dt} \theta(t) = -\gamma_n \cdot k \cdot (u_c(t))^2 \cdot (\theta(t) - \theta^0) \]
MIT rule (Example 5.3), cont’d

For the system and model given by

\[ y(t) = k \cdot u(t), \quad y_m(t) = k_0 \cdot u_c(t) \]

we define \( e(t) = y(t) - y_m(t) \) and take

\[ u(t) = \theta(t) u_c(t), \quad \frac{d}{dt} \theta(t) = -\gamma_n \cdot u_c(t) \cdot e(t) \]
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\[ u(t) = \theta(t) u_c(t), \quad \frac{d}{dt} \theta(t) = -\gamma_n \cdot u_c(t) \cdot e(t) \]
As the result we obtain
\[ \theta(t) = \theta^0 + \sigma(t), \quad e(t) = k \cdot \sigma(t) \cdot u_c(t) \]
\[ \sigma(t) = \exp\left\{ -\gamma_n \cdot k \cdot I_t \right\} (\theta(0) - \theta^0), \quad I_t = \int_0^t (u_c(t))^2 \, d\tau \]
MIT rule (Example 5.3), cont’d

For the system and model given by

\[ y(t) = k \cdot u(t), \quad y_m(t) = k_0 \cdot u_c(t) \]

we define \( e(t) = y(t) - y_m(t) \) and take

\[ u(t) = \theta(t) u_c(t), \quad \frac{d}{dt} \theta(t) = -\gamma_n \cdot u_c(t) \cdot e(t) \]

As the result we obtain

\[ \theta(t) = \theta^0 + \sigma(t), \quad e(t) = k \cdot \sigma(t) \cdot u_c(t) \]

\[ \sigma(t) = \exp \left\{ -\gamma_n \cdot k \cdot I_t \right\} \left( \theta(0) - \theta^0 \right), \quad I_t = \int_0^t (u_c(t))^2 \, d\tau \]

If \( \theta(0) \neq \theta^0 \), for \( e(t) \to 0 \) as \( t \to \infty \) we need:

\[ \exp \left\{ -\gamma_n \cdot k \cdot I_t \right\} \to 0 \quad \text{or} \quad u_c(t) \to 0 \]
For the system and model given by
\[ y(t) = k \cdot u(t), \quad y_m(t) = k_0 \cdot u_c(t) \]
we define \( e(t) = y(t) - y_m(t) \) and take
\[ u(t) = \theta(t) \ u_c(t), \quad \frac{d}{dt} \theta(t) = -\gamma_n \cdot u_c(t) \cdot e(t) \]
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\[ \theta(t) = \theta^0 + \sigma(t), \quad e(t) = k \cdot \sigma(t) \cdot u_c(t) \]
\[ \sigma(t) = \exp \left\{ -\gamma_n \cdot k \cdot I_t \right\} (\theta(0) - \theta^0), \quad I_t = \int_0^t (u_c(t))^2 \, d\tau \]
If \( \theta(0) \neq \theta^0 \), for \( e(t) \rightarrow 0 \) as \( t \rightarrow \infty \) we need:
\[ I_t = \int_0^t (u_c(t))^2 \, d\tau \rightarrow \infty \quad \text{or} \quad u_c(t) \rightarrow 0 \]
Tuning the Gain for MIT rule

Consider again the problem with scaling the reference

\[ y = k \cdot G(p) \, u, \quad y_m = k_0 \cdot G(p) \, u_c, \quad u = \theta \, u_c \]

where \( \theta(t) \) is determined by MIT rule:

\[ \frac{d}{dt} \theta = -\gamma \cdot y_m \cdot e, \quad e = y - y_m \]
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\[ \frac{d}{dt} \theta = -\gamma \cdot y_m \cdot e, \quad e = y - y_m \]

The equation for \( \theta \) can be re-written as follows

\[ \frac{d}{dt} \theta = -\gamma \cdot y_m \cdot (y - y_m) = -\gamma \cdot y_m \cdot (k \cdot G(p) \cdot u_c - y_m) \]
Tuning the Gain for MIT rule

Consider again the problem with scaling the reference

\[ y = k \cdot G(p) \cdot u, \quad y_m = k_0 \cdot G(p) \cdot u_c, \quad u = \theta \cdot u_c \]

where \( \theta(t) \) is determined by MIT rule:

\[ \frac{d}{dt} \theta = -\gamma \cdot y_m \cdot e, \quad e = y - y_m \]

The equation for \( \theta \) can be re-written as follows

\[ \frac{d}{dt} \theta(t) + \gamma \cdot k \cdot y_m(t) \cdot G(p) \left[ \theta(t) u_c(t) \right] = \gamma y_m^2(t) \]
Tuning the Gain for MIT rule

Consider again the problem with scaling the reference

\[ y = k \cdot G(p) \cdot u, \quad y_m = k_0 \cdot G(p) \cdot u_c, \quad u = \theta \cdot u_c \]

where \( \theta(t) \) is determined by MIT rule:

\[
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The equation for \( \theta \) can be re-written as follows

\[
\frac{d}{dt} \theta(t) + \gamma \cdot k \cdot y_m(t) \cdot G(p) \left[ \theta(t) u_c(t) \right] = \gamma y_m^2(t)
\]

Here

- the functions \( y_m(t) \) and \( u_c(t) \) are known!
- the range of the constant gain \( \gamma \), for which the nominal value \( \theta^0 \) (its stationary point) is stable, should be determined.
Tuning the Gain for MIT rule (cont’d)

\[
\frac{d}{dt} \theta(t) + \gamma \cdot k \cdot y_m(t) \cdot G(p) \left[ \theta(t) u_c(t) \right] = \gamma y_m^2(t)
\]

In general the analysis of stability is difficult!
Tuning the Gain for MIT rule (cont’d)

\[
\frac{d}{dt} \theta(t) + \gamma \cdot k \cdot y_m(t) \cdot G(p) \left[ \theta(t) u_c(t) \right] = \gamma y_m^2(t)
\]

In general the analysis of stability is difficult!

Consider the case when \( y_m(t) \equiv y_m^o, u_c(t) = u_c^o \), then ODE

\[
\frac{d}{dt} \theta(t) + \gamma \cdot k \cdot y_m^o \cdot u_c^o \cdot G(p) \left[ \theta(t) \right] = \gamma (y_m^o)^2
\]

is linear and time-invariant!
Tuning the Gain for MIT rule (cont’d)

\[
\frac{d}{dt} \theta(t) + \gamma \cdot k \cdot y_m(t) \cdot G(p) \left[ \theta(t) \ u_c(t) \right] = \gamma y_m^2(t)
\]

In general the analysis of stability is difficult!

Consider the case when \( y_m(t) \equiv y_m^0, u_c(t) = u_c^0 \), then ODE

\[
\frac{d}{dt} \theta(t) + \gamma \cdot k \cdot y_m^0 \cdot u_c^0 \cdot G(p) \left[ \theta(t) \right] = \gamma (y_m^0)^2
\]

is linear and time-invariant!

Stability is determined by the roots of the algebraic equation

\[
s + \mu \cdot G(s) = 0, \quad \mu = \gamma \cdot k \cdot y_m^0 \cdot u_c^0
\]
Tuning the Gain for MIT rule (cont’d)

\[
\frac{d}{dt}\theta(t) + \gamma \cdot k \cdot y_m(t) \cdot G(p) \left[ \theta(t) u_c(t) \right] = \gamma y_m^2(t)
\]

In general the analysis of stability is difficult!

Consider the case when \( y_m(t) \equiv y_m^o \), \( u_c(t) = u_c^o \), then ODE

\[
\frac{d}{dt}\theta(t) + \gamma \cdot k \cdot y_m^o \cdot u_c^0 \cdot G(p) \left[ \theta(t) \right] = \gamma (y_m^o)^2
\]

is linear and time-invariant!

Stability is determined by the roots of the algebraic equation

\[
s + \mu \cdot G(s) = 0, \quad \mu = \gamma \cdot k \cdot y_m^o \cdot u_c^0
\]

Root locus analysis (variation of zeros with \( \mu \)) can be used. A reasonable value for \( \gamma \) can be obtained from this analysis and might work for slowly varying signals.
Example 5.4

Let, as in Example 5.1

\[ G(s) = \frac{1}{s + 1}, \quad k = 1, \quad k_0 = 2 \]
Example 5.4

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The characteristic equation

\[ s + \mu \frac{1}{s + 1} = 0 \iff s^2 + s + \mu = 0 \]

has stable zeros if and only if

\[ \mu = \gamma \cdot k \cdot y_m^0 \cdot u_c^0 = \gamma \cdot \left( k_0 \cdot G(0) \cdot u_c^0 \right) \cdot u_c^0 = 2 \gamma (u_c^0)^2 > 0 \]
**Example 5.4**

Let, as in Example 5.1

\[ G(s) = \frac{1}{s + 1}, \quad k = 1, \quad k_0 = 2 \]

The characteristic equation

\[ s + \mu \frac{1}{s + 1} = 0 \quad \iff \quad s^2 + s + \mu = 0 \]

has stable zeros if and only if

\[ \mu = \gamma \cdot k \cdot y_m^0 \cdot u_c^0 = \gamma \cdot \left( k_0 G(0) u_c^0 \right) \cdot u_c^0 = 2 \gamma \left( u_c^0 \right)^2 > 0 \]

So, \( \gamma > 0 \) will work.

Note, however, that the transient depends on \( u_c^0 \)!

The relative damping is \( \zeta = \frac{1}{2 \sqrt{\mu}} = \frac{1}{2 \sqrt{2 \gamma |u_c^0|}} \).

\( \mu \approx 1 \) is reasonable \( \iff \) take \( \gamma \approx 0.5 \) for \( u_c^0 \approx 1 \) in average.
Example 5.5

Consider the stable system with relative degree 2:

\[
G(s) = \frac{1}{s^2 + a_1 s + a_2}, \quad a_1 > 0, \quad a_2 > 0
\]
Example 5.5

Consider the stable system with relative degree 2:

\[ G(s) = \frac{1}{s^2 + a_1 s + a_2}, \quad a_1 > 0, \quad a_2 > 0 \]

The characteristic equation

\[ s + \mu \frac{1}{s^2 + a_1 s + a_2} = 0 \iff s^3 + a_1 s^2 + a_2 s + \mu = 0 \]

has stable zeros if and only if

\[ \mu > 0 \quad \text{and} \quad a_1 a_2 > \mu = \gamma \cdot k \cdot y_m^o \cdot u_c^0 \]
Example 5.5

Consider the stable system with relative degree 2:

\[ G(s) = \frac{1}{s^2 + a_1 s + a_2}, \quad a_1 > 0, \quad a_2 > 0 \]

The characteristic equation

\[ s + \mu \frac{1}{s^2 + a_1 s + a_2} = 0 \iff s^3 + a_1 s^2 + a_2 s + \mu = 0 \]

has stable zeros if and only if

\[ \mu > 0 \quad \text{and} \quad a_1 a_2 > \mu = \gamma \cdot k \cdot y_m^o \cdot u_c^0 \]

Conclusion: with any choice of \( \gamma > 0 \), stability is lost for sufficiently large magnitudes of the reference signal \( u_c^0 \).
Figure 5.8  Simulation of the MRAS in Example 5.5. The command signal is a square wave with the amplitude (a) 0.1, (b) 1, and (c) 3.5. The model output $y_m$ is a dashed line; the process output is a solid line. The following parameters are used: $k = a_1 = a_2 = \theta^0 = 1$, and $\gamma = 0.1$. 
Normalized MIT rule

\[
\frac{d}{dt} \theta = -\gamma \cdot e(t, \theta) \cdot \frac{\phi}{\alpha + \phi^T \phi}, \quad \phi = \frac{\partial}{\partial \theta} e(t, \theta), \quad \alpha > 0
\]

**Figure 5.9** Simulation of the MRAS in Example 5.5 with the normalized MIT rule. The command signal is a square wave with the amplitude (a) 0.1, (b) 1, and (c) 3.5. Compare with Fig. 5.8. The model output \( y_m \) is a dashed line; the process output is a solid line. The parameters used are \( k = a_1 = a_2 = \theta^p = 1, \alpha = 0.001 \), and \( \gamma = 0.1 \).
Next Lecture / Assignments:

Next meeting (May 24, 13:00-15:00, in A208Tekn):
Lyapunov-based design.

Homework problem: The process and model are described by
\[ G(s) = \frac{1}{s}, \quad G_m(s) = \frac{2}{s + 2} \]

For the control law
\[ u(t) = \theta_1 u_c(t) - \theta_2 y(t) \]

design an MIT-like adaptation law such that
\[ \theta_i \approx -\left(\gamma_1 + \gamma_2 \frac{1}{p}\right) e \frac{\partial}{\partial \theta_i} e, \quad i \in \{1, 2\}. \]

Simulate the MRAS with various gains.
Consider \( \gamma_{1,2} \in \{0, 1, 5\} \) and a unit square wave for \( u_c(t) \).
Compare performance for different combinations.