Lecture 16: Geometric Nonlinear Control

- Background and Useful Concepts
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- Background and Useful Concepts
- Feedback Linearization
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- Background and Useful Concepts
- Feedback Linearization
- Feedback Linearization for Single Link Flexible Joint
A smooth manifold $\mathcal{M}$ of dimension $m$ is a subset of $\mathbb{R}^n$ defined as the zero set of $p = (n - m)$ smooth scalar functions

$$h_1(x_1, x_2, \ldots, x_n) = 0$$
$$\vdots$$
$$h_p(x_1, x_2, \ldots, x_n) = 0$$

such that the differentials

$$dh_1(\cdot), \quad dh_2(\cdot), \quad \ldots, \quad dh_p(\cdot)$$

are linear independent for any point of $\mathcal{M}$. 

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A Smooth Manifold in $\mathbb{R}^n$

A smooth manifold $\mathcal{M}$ of dimension $m$ is a subset of $\mathbb{R}^n$ defined as the zero set of $p = (n - m)$ smooth scalar functions

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such that the differentials

$$dh_1(\cdot), \quad dh_2(\cdot), \quad \ldots, \quad dh_p(\cdot)$$

are linear independent for any point of $\mathcal{M}$.

At each point $x \in \mathcal{M}$ we can attach a tangent space $T_x\mathcal{M}$, which is an $m$-dimensional space that specifies the set of possible directions of velocities at this point.
Example

Consider a unit sphere $S^2$ in $\mathbb{R}^3$ defined by

$$h(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$$

What is the dimension of this manifold? Is it smooth?
Example

The differential $dh(\cdot)$ of the function

$$h(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$$

at the point $(x_0, y_0, z_0) \in S^2$ is the row vector

$$dh(x_0, y_0, z_0) = \left[ \frac{\partial}{\partial x} h, \frac{\partial}{\partial y} h, \frac{\partial}{\partial z} h \right] \bigg|_{x=x_0, y=y_0, z=z_0}$$
Example

The differential $dh(\cdot)$ of the function

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Example

The differential $dh(\cdot)$ of the function

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How to compute a tangent space $T_{p_0}S^2$ to the sphere $S^2$ at the point $p_0 = (x_0, y_0, z_0) \in S^2$?
Example

The differential \( dh(\cdot) \) of the function

\[
h(x, y, z) = x^2 + y^2 + z^2 - 1 = 0
\]

at the point \((x_0, y_0, z_0) \in S^2\) is the row vector

\[
dh(x_0, y_0, z_0) = \left[ \frac{\partial}{\partial x} h, \frac{\partial}{\partial y} h, \frac{\partial}{\partial z} h \right] \bigg|_{x=x_0, y=y_0, z=z_0} = [2x_0, 2y_0, 2z_0]
\]

By definition

\[
T_{p_0} S^2 = \left\{ V = [v_1, v_2, v_3]^T : dh(p_0) \perp V \right\}
\]

\[
= \left\{ V = [v_1, v_2, v_3]^T : 2x_0 \cdot v_1 + 2y_0 \cdot v_2 + 2z_0 \cdot v_3 = 0 \right\}
\]
Example

The differential $dh(\cdot)$ of the function

$$h(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$$

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$$= \left\{ V = [v_1, v_2, v_3]^T : 2x_0 \cdot v_1 + 2y_0 \cdot v_2 + 2z_0 \cdot v_3 = 0 \right\}$$

$$\vec{V}_1 = [1, 0, -x_0/z_0], \quad \vec{V}_2 = [0, 1, -y_0/z_0]$$
A smooth vector field on a manifold $\mathcal{M}$ is a smooth function

$$x \in \mathcal{M} \rightarrow f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix} \in T_x \mathcal{M}$$
Smooth Vector and Co-vector Fields

A smooth vector field on a manifold $\mathcal{M}$ is a smooth function

$$x \in \mathcal{M} \rightarrow f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix} \in T_x \mathcal{M}$$

A smooth co-vector field on a manifold $\mathcal{M}$ is a smooth function

$$x \in \mathcal{M} \rightarrow w(x) = [w_1(x), \cdots, w_m(x)] \in T^*_x \mathcal{M}$$

Here $T^*_x \mathcal{M}$ is co-tangent space to the manifold $\mathcal{M}$ at its point $x$. 
Distributions and Co-distributions

Let $X_1(x), \ldots, X_k(x)$ be vector fields on $\mathcal{M}$ that are linearly independent for each $x \in \mathcal{M}$. Then a \textbf{distribution $\triangle(\cdot)$} is a family of linear subspaces of $T\mathcal{M}$ defined for each point $x$ as

$$\triangle(x) = \text{span}\ \{X_1(x), \ldots, X_k(x)\}$$
Distributions and Co-distributions

Let $X_1(x), \ldots, X_k(x)$ be vector fields on $\mathcal{M}$ that are linearly independent for each $x \in \mathcal{M}$. Then a distribution $\mathcal{D}(\cdot)$ is a family of linear subspaces of $T\mathcal{M}$ defined for each point $x$ as

$$\mathcal{D}(x) = \text{span} \{ X_1(x), \ldots, X_k(x) \}$$

Let $W_1(x), \ldots, W_k(x)$ be co-vector fields on $\mathcal{M}$ linearly independent for each $x \in \mathcal{M}$. Then a co-distribution $\mathcal{W}(\cdot)$ is a family of linear subspaces of $T^*\mathcal{M}$ defined for each point $x$ as

$$\mathcal{W}(x) = \text{span} \{ W_1(x), \ldots, W_k(x) \}$$
Distributions and Co-distributions

Let $X_1(x), \ldots, X_k(x)$ be vector fields on $\mathcal{M}$ that are linearly independent for each $x \in \mathcal{M}$. Then a distribution $\triangle(\cdot)$ is a family of linear subspaces of $T\mathcal{M}$ defined for each point $x$ as

$$\triangle(x) = \text{span}\left\{X_1(x), \ldots, X_k(x)\right\}$$

Let $W_1(x), \ldots, W_k(x)$ be co-vector fields on $\mathcal{M}$ linearly independent for each $x \in \mathcal{M}$. Then a co-distribution $\Omega(\cdot)$ is a family of linear subspaces of $T^*\mathcal{M}$ defined for each point $x$ as

$$\Omega(x) = \text{span}\left\{W_1(x), \ldots, W_k(x)\right\}$$

$$\frac{d}{dt} x = f(x) + g_1(x)u_1 + \cdots + g_m(x)u_m$$
Lie Bracket of Vector Fields

Given two vector fields \( f(\cdot) \) and \( g(\cdot) \) on \( \mathcal{M} = \mathbb{R}^n \).

The **Lie bracket** of \( f \) and \( g \) is a new vector field in \( \mathbb{R}^n \) denoted by \([f, g]\) and defined for each \( x \in \mathcal{M} = \mathbb{R}^n \) by

\[
[f, g](x) := \left[ \frac{\partial}{\partial x} g(x) \right] f(x) - \left[ \frac{\partial}{\partial x} f(x) \right] g(x)
\]
Lie Bracket of Vector Fields

Given two vector fields \( f(\cdot) \) and \( g(\cdot) \) on \( \mathcal{M} = \mathbb{R}^n \).

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\[
[f, g](x) := \left[ \frac{\partial}{\partial x} g(x) \right] f(x) - \left[ \frac{\partial}{\partial x} f(x) \right] g(x)
\]

Suppose \( f = Ax \) and \( g = Bx \), then

\[
[f, g](x) = \left[ \frac{\partial}{\partial x} g(x) \right] f(x) - \left[ \frac{\partial}{\partial x} f(x) \right] g(x)
\]

\[
= B \cdot Ax - A \cdot Bx = (BA - AB)x
\]

\[
= C
\]
Lie Bracket of Vector Fields

Given two vector fields $f(\cdot)$ and $g(\cdot)$ on $\mathcal{M} = \mathbb{R}^n$.

The **Lie bracket** of $f$ and $g$ is a new vector field in $\mathbb{R}^n$ denoted by $[f, g]$ and defined for each $x \in \mathcal{M} = \mathbb{R}^n$ by

$$[f, g](x) := \left[ \frac{\partial}{\partial x} g(x) \right] f(x) - \left[ \frac{\partial}{\partial x} f(x) \right] g(x)$$

**Notations:**

$$ad^0_f (g) = g$$
$$ad^1_f (g) = [f, g]$$
$$ad^2_f (g) = [f, [f, g]]$$
$$\vdots$$
$$ad^k_f (g) = [f, ad^{(k-1)}_f (g)]$$
Lie Derivative of Scalar Function

Given

- a scalar function $h(\cdot)$ on $\mathbb{R}^n$, i.e. $h : \mathbb{R}^n \to \mathbb{R}$
- a vector field $f(\cdot)$ on $\mathbb{R}^n$, i.e. $f : \mathbb{R}^n \to \mathbb{R}^n$

The Lie derivative of $h(\cdot)$ with respect to $f(\cdot)$ is a new scalar function denoted as $\mathcal{L}_f h$ and defined by

$$\mathcal{L}_f h = \left[ \frac{\partial}{\partial x} h(x) \right] f(x) = \left[ \frac{\partial}{\partial x_1} h(x) \right] f_1(x) + \cdots + \left[ \frac{\partial}{\partial x_n} h(x) \right] f_n(x)$$
Lie Derivative of Scalar Function

Given

- a scalar function $h(\cdot)$ on $\mathbb{R}^n$, i.e. $h : \mathbb{R}^n \to \mathbb{R}$
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Recursive notation:

$$\mathcal{L}^k_f h = \mathcal{L}_f \left[ \mathcal{L}^{(k-1)}_f h \right], \quad k = 1, 2, \ldots, \quad \mathcal{L}_f^0 h = h$$
Lie Derivative of Scalar Function

Given

- a scalar function $h(\cdot)$ on $\mathbb{R}^n$, i.e. $h : \mathbb{R}^n \rightarrow \mathbb{R}$
- a vector field $f(\cdot)$ on $\mathbb{R}^n$, i.e. $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$

The Lie derivative of $h(\cdot)$ with respect to $f(\cdot)$ is a new scalar function denoted as $\mathcal{L}_f h$ and defined by

$$\mathcal{L}_f h = \left[ \frac{\partial}{\partial x} h(x) \right] f(x) = \left[ \frac{\partial}{\partial x_1} h(x) \right] \cdot f_1(x) + \cdots + \left[ \frac{\partial}{\partial x_n} h(x) \right] \cdot f_n(x)$$

Lemma: Let $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be a scalar function and $f$, $g$ be vector fields on $\mathbb{R}^n$. Then the following identity holds

$$\mathcal{L}_{[f,g]} h \equiv \mathcal{L}_f [\mathcal{L}_g h] - \mathcal{L}_g [\mathcal{L}_f h]$$
Frobenius Theorem

Consider two partial differential equations

\[
\frac{\partial z}{\partial x} = f(x, y, z), \quad \frac{\partial z}{\partial y} = g(x, y, z)
\]
Frobenius Theorem

Consider two partial differential equations

\[
\frac{\partial z}{\partial x} = f(x, y, z), \quad \frac{\partial z}{\partial y} = g(x, y, z)
\]

We are looking for a scalar variable \( z \) in the form

\[
z = \phi(x, y)
\]

such that

\[
\frac{\partial}{\partial x} \phi(x, y) = f(x, y, \phi(x, y)), \quad \frac{\partial}{\partial y} \phi(x, y) = g(x, y, \phi(x, y))
\]
Frobenius Theorem

Consider two partial differential equations

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\frac{\partial z}{\partial x} = f(x, y, z), \quad \frac{\partial z}{\partial y} = g(x, y, z)
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\]

If we find the solution, then it corresponds to the surface in \( \mathbb{R}^3 \)

\[
\mathcal{M} = \{ x, y, \quad z = \phi(x, y) \}
\]

How to compute a tangent plane to that surface (manifold)?
**Example (Sphere in $\mathbb{R}^3$)**

The differential $dh(\cdot)$ of the function

$$h(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$$

at the point $(x_0, y_0, z_0) \in S^2$ is the row vector

$$dh(x_0, y_0, z_0) = \left[ \frac{\partial}{\partial x} h, \frac{\partial}{\partial y} h, \frac{\partial}{\partial x} h \right] \bigg|_{x=x_0, y=y_0, z=z_0}$$

$$= [2x_0, 2y_0, 2z_0]$$

By definition

$$T_{p_0} S^2 = \{ V = [v_1, v_2, v_3]^T : dh(p_0) \perp V \}$$

$$= \left\{ V = [v_1, v_2, v_3]^T : 2x_0 \cdot v_1 + 2y_0 \cdot v_2 + 2z_0 \cdot v_3 = 0 \right\}$$

$$\vec{V}_1 = [1, 0, -x_0/z_0], \quad \vec{V}_2 = [0, 1, -y_0/z_0]$$
Computing a Tangent Space

The differential $dh(\cdot)$ of the function

$$h(x, y, z) = z - \phi(x, y) = 0$$

at the point $p_0 = (x_0, y_0, z_0) \in \mathbb{R}^2$ is the row vector

$$dh(p_0) = \left[ \frac{\partial}{\partial x} h, \frac{\partial}{\partial y} h, \frac{\partial}{\partial z} h \right] \bigg|_{x=x_0, y=y_0, z=z_0}$$

$$= \left[ - \frac{\partial}{\partial x} \phi(x, y), - \frac{\partial}{\partial x} \phi(x, y), 1 \right] \bigg|_{x=x_0, y=y_0, z=z_0}$$
Computing a Tangent Space

The differential $dh(\cdot)$ of the function

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at the point $p_0 = (x_0, y_0, z_0) \in \mathbb{R}^2$ is the row vector

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$$= \left[ -\frac{\partial}{\partial x} \phi(x, y), -\frac{\partial}{\partial y} \phi(x, y), 1 \right] \bigg|_{x=x_0, y=y_0, z=z_0}$$

$$= \left[ -f(x_0, y_0, \phi(x_0, y_0)), -g(x_0, y_0, \phi(x_0, y_0)), 1 \right]$$
Computing a Tangent Space

The differential \( dh(\cdot) \) of the function

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h(x, y, z) = z - \phi(x, y) = 0
\]

at the point \( p_0 = (x_0, y_0, z_0) \in \mathbb{R}^2 \) is the row vector

\[
dh(p_0) = \left[ \frac{\partial}{\partial x} h, \frac{\partial}{\partial y} h, \frac{\partial}{\partial x} h \right]_{x=x_0, y=y_0, z=z_0}
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= \left[ -\frac{\partial}{\partial x} \phi(x, y), -\frac{\partial}{\partial y} \phi(x, y), 1 \right]_{x=x_0, y=y_0, z=z_0}
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\[
= \left[ -f(x_0, y_0, \phi(x_0, y_0)), -g(x_0, y_0, \phi(x_0, y_0)), 1 \right]
\]

\[
T_{p_0} M = \left\{ \vec{V} = [v_1, v_2, v_3]^T : dh(p_0) \perp \vec{V} \right\}
\]

\[
= \left\{ \vec{V} : -f(\cdot)v_1 - g(\cdot)v_2 + 1 \cdot v_3 = 0 \right\}
\]

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Computing a Tangent Space

The differential \( dh(\cdot) \) of the function

\[
    h(x, y, z) = z - \phi(x, y) = 0
\]

at the point \( p_0 = (x_0, y_0, z_0) \in \mathbb{R}^2 \) is the row vector

\[
    dh(p_0) = \begin{bmatrix}
        \frac{\partial}{\partial x} h, & \frac{\partial}{\partial y} h, & \frac{\partial}{\partial x} h
    \end{bmatrix}
    \bigg|_{x=x_0, y=y_0, z=z_0}
\]

\[
    = \begin{bmatrix}
        -\frac{\partial}{\partial x} \phi(x, y), & -\frac{\partial}{\partial y} \phi(x, y), & 1
    \end{bmatrix}
    \bigg|_{x=x_0, y=y_0, z=z_0}
\]

\[
    = \begin{bmatrix}
        -f(x_0, y_0, \phi(x_0, y_0)), & -g(x_0, y_0, \phi(x_0, y_0)), & 1
    \end{bmatrix}
\]

\[
    T_{p_0}M = \left\{ \vec{V} = [v_1, v_2, v_3]^T : dh(p_0) \perp \vec{V} \right\}
\]

\[
    = \left\{ \vec{V} : -f(\cdot)v_1 - g(\cdot)v_2 + 1 \cdot v_3 = 0 \right\}
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Computing a Tangent Space

The differential $dh(\cdot)$ of the function

\[ h(x, y, z) = z - \phi(x, y) = 0 \]

at the point $p_0 = (x_0, y_0, z_0) \in \mathbb{R}^2$ is the row vector

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dh(p_0) = \left[ \frac{\partial}{\partial x} h, \frac{\partial}{\partial y} h, \frac{\partial}{\partial x} h \right] \bigg|_{x=x_0, y=y_0, z=z_0}
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\[
= \left[ -f(x_0, y_0, \phi(x_0, y_0)), -g(x_0, y_0, \phi(x_0, y_0)), 1 \right]
\]

\[
T_{p_0} \mathcal{M} = \left\{ \vec{V} : -f(\cdot)v_1 - g(\cdot)v_2 + 1 \cdot v_3 = 0 \right\}
\]

Basis for $T_{p_0} \mathcal{M}$ : $\vec{V}_1 = [1, 0, f(\cdot)]^T$, $\vec{V}_2 = [0, 1, g(\cdot)]^T$
**Problem Reformulation**

Searching for solution \( z = \phi(x, y) \) of two partial differential equations

\[
\frac{\partial}{\partial x} z = f(x, y, z), \quad \frac{\partial}{\partial y} z = g(x, y, z)
\]

is equivalent to search

Surface \( M \) in \( \mathbb{R}^3 \) whose tangent space in each point \( T_pM \) is spanned by \( \vec{V}_1 = [1, 0, f(x, y, z)]^T \) and \( \vec{V}_2 = [0, 1, g(x, y, z)]^T \)
Problem Reformulation

Searching for solution \( z = \phi(x, y) \) of two partial differential equations

\[
\frac{\partial}{\partial x} z = f(x, y, z), \quad \frac{\partial}{\partial y} z = g(x, y, z)
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Surface \( \mathcal{M} \) in \( \mathbb{R}^3 \) whose tangent space in each point \( T_p \mathcal{M} \) is spanned by

\[
\vec{V}_1 = [1, 0, f(x, y, z)]^T \quad \text{and} \quad \vec{V}_2 = [0, 1, g(x, y, z)]^T
\]

If such surface \( \mathcal{M} \) can be found then

\( \mathcal{M} \) is integral manifold of the system
Problem Reformulation

Searching for solution \( z = \phi(x, y) \) of two partial differential equations

\[
\frac{\partial}{\partial x} z = f(x, y, z), \quad \frac{\partial}{\partial y} z = g(x, y, z)
\]

is equivalent to search

Surface \( \mathcal{M} \) in \( \mathbb{R}^3 \) whose tangent space in each point \( T_p \mathcal{M} \) is spanned by \( \vec{V}_1 = \begin{bmatrix} 1, 0, f(x, y, z) \end{bmatrix}^T \) and \( \vec{V}_2 = \begin{bmatrix} 0, 1, g(x, y, z) \end{bmatrix}^T \)

If such surface \( \mathcal{M} \) can be found then \( \mathcal{M} \) is integral manifold of the system

If such surface \( \mathcal{M} \) can be found then the equations are called integrable
Problem Reformulation

Search for solution \( z = \phi(x, y) \) of equations

\[
\frac{\partial}{\partial x} z = f(x, y, z), \quad \frac{\partial}{\partial y} z = g(x, y, z)
\]

can be rewritten as the search for solution of other two equations

\[
\mathcal{L}_{\vec{V}_1} h = 0, \quad \mathcal{L}_{\vec{V}_2} h = 0
\]

where \( h = h(x, y, z) \) and

\[
\vec{V}_1 = [1, 0, f(x, y, z)]^T \quad \text{and} \quad \vec{V}_2 = [0, 1, g(x, y, z)]^T
\]
Problem Reformulation

Search for solution \( z = \phi(x, y) \) of equations

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\frac{\partial}{\partial x} z = f(x, y, z), \quad \frac{\partial}{\partial y} z = g(x, y, z)
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can be rewritten as the search for solution of other two equations

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\mathcal{L}_{\vec{V}_1} h = 0, \quad \mathcal{L}_{\vec{V}_2} h = 0
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where \( h = h(x, y, z) \) and

\[
\vec{V}_1 = [1, 0, f(x, y, z)]^T \quad \text{and} \quad \vec{V}_2 = [0, 1, g(x, y, z)]^T
\]

Indeed, if the solution \( z = \phi(x, y) \) exists, then the equation

\[
\mathcal{L}_{\vec{V}_1} h = \left[ \frac{\partial}{\partial x} h(x, y, z) \right] \cdot 1 + \left[ \frac{\partial}{\partial y} h(x, y, z) \right] \cdot 0 + \\
\quad + \left[ \frac{\partial}{\partial z} h(x, y, z) \right] \cdot f(x, y, z) = 0
\]
Problem Reformulation

Search for solution $z = \phi(x, y)$ of equations

$$\frac{\partial}{\partial x} z = f(x, y, z), \quad \frac{\partial}{\partial y} z = g(x, y, z)$$

can be rewritten as the search for solution of other two equations

$$\mathcal{L}_{\vec{V}_1} h = 0, \quad \mathcal{L}_{\vec{V}_2} h = 0$$

where $h = h(x, y, z)$ and

$$\vec{V}_1 = [1, 0, f(x, y, z)]^T \quad \text{and} \quad \vec{V}_2 = [0, 1, g(x, y, z)]^T$$

Indeed, if the solution $z = \phi(x, y)$ exists, then the equation

$$\mathcal{L}_{\vec{V}_1} h = \left[ \frac{\partial}{\partial x} h(x, y, z) \right] \cdot 1 + \left[ \frac{\partial}{\partial z} h(x, y, z) \right] \cdot f(x, y, z) = 0$$

has a solution $h(x, y, z) = z - \phi(x, y)$
Problem Reformulation

Search for solution $z = \phi(x, y)$ of equations

$$\frac{\partial}{\partial x} z = f(x, y, z), \quad \frac{\partial}{\partial y} z = g(x, y, z)$$

can be rewritten as the search for solution of other two equations

$$\mathcal{L}_{\vec{V}_1} h = 0, \quad \mathcal{L}_{\vec{V}_2} h = 0$$

where $h = h(x, y, z)$ and

$$\vec{V}_1 = [1, 0, f(x, y, z)]^T \quad \text{and} \quad \vec{V}_2 = [0, 1, g(x, y, z)]^T$$

Indeed, if the solution $z = \phi(x, y)$ exists, then the equation

$$\mathcal{L}_{\vec{V}_2} h = \left[ \frac{\partial}{\partial x} h(x, y, z) \right] \cdot 0 + \left[ \frac{\partial}{\partial y} h(x, y, z) \right] \cdot 1 +$$

$$+ \left[ \frac{\partial}{\partial z} h(x, y, z) \right] \cdot g(x, y, z) = 0$$
Problem Reformulation

Search for solution \( z = \phi(x, y) \) of equations

\[
\frac{\partial}{\partial x} z = f(x, y, z), \quad \frac{\partial}{\partial y} z = g(x, y, z)
\]

can be rewritten as the search for solution of other two equations

\[
\mathcal{L}_{\vec{V}_1} h = 0, \quad \mathcal{L}_{\vec{V}_2} h = 0
\]

where \( h = h(x, y, z) \) and

\[
\vec{V}_1 = [1, 0, f(x, y, z)]^T \quad \text{and} \quad \vec{V}_2 = [0, 1, g(x, y, z)]^T
\]

Indeed, if the solution \( z = \phi(x, y) \) exists, then the equation

\[
\mathcal{L}_{\vec{V}_2} h = \left[ \frac{\partial}{\partial y} h(x, y, z) \right] \cdot 1 + \left[ \frac{\partial}{\partial z} h(x, y, z) \right] \cdot g(x, y, z) = 0
\]

has a solution \( h(x, y, z) = z - \phi(x, y) \)
Concepts of Integrability and Involutivity for Distributions

The distribution

$$\triangle = \text{span} \{X_1(x), X_2(x), \ldots, X_m(x)\}, \quad x \in \mathbb{R}^n$$

is said to be **completely integrable**, if there are functions $h_1(x), \ldots, h_{n-m}(x)$ such that

- they are linearly independent for any $x \in \mathbb{R}^n$;
- they satisfy the system of partial differential equations

$$\mathcal{L}_{X_i} h_j = 0, \quad \forall i \in \{1, \ldots, m\}, \forall j \in \{1, \ldots, (n-m)\}$$
Concepts of Integrability and Involutivity for Distributions

The distribution

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- they are linearly independent for any $x \in \mathbb{R}^n$;
- they satisfy the system of partial differential equations

$$\mathcal{L}_{X_i}h_j = 0, \quad \forall i \in \{1, \ldots, m\}, \forall j \in \{1, \ldots, (n-m)\}$$

The distribution $\triangle$ is said to be **involutive**, if there are scalar functions $\alpha_{ijk} : \mathbb{R}^n \to \mathbb{R}$ such that

$$[X_i, X_j] = \sum_{k=1}^{n} \alpha_{ijk}X_k \quad \forall i, j$$
A distribution \( \Delta \) on \( \mathbb{R}^n \) is integrable

\[ \iff \]

A distribution \( \Delta \) on \( \mathbb{R}^n \) is involutive
Lecture 16: Geometric Nonlinear Control

- Background and Useful Concepts
- Feedback Linearization
- Feedback Linearization for Single Link Flexible Joint
Concept of Feedback Linearization

The control system

\[ \frac{d}{dt} x = f(x) + g(x)u, \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^1, \ f(0) = 0 \]

is said to be feedback linearizable if

- there is a invertible change of coordinates \( y = T(x) \)
- there is a feedback transform \( u = \alpha(x) + \beta(x)v \)

such that the system dynamics written in new coordinates is

\[ \frac{d}{dt} y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} v \]

\[ = A y + B v \]
How to find $T(\cdot)$, $\alpha(\cdot)$ and $\beta(\cdot)$?

If the system
\[
\frac{d}{dt}x = f(x) + g(x)u, \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^1, \ f(0) = 0
\]

is feedback linearizable, then there are new coordinates $y$ defined by
\[
y = T(x) \quad \Rightarrow \quad \frac{d}{dt}y = \frac{d}{dt} [T(x)] = \frac{\partial}{\partial x} [T(x)] \frac{d}{dt}x
\]
How to find $T(\cdot), \alpha(\cdot)$ and $\beta(\cdot)$?

If the system

$$\frac{d}{dt}x = f(x) + g(x)u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^1, \quad f(0) = 0$$

is feedback linearizable, then there are new coordinates $y$ defined by

$$y = T(x) \quad \Rightarrow \quad \frac{d}{dt}y = \frac{d}{dt}[T(x)] = \frac{\partial}{\partial x}[T(x)] \frac{d}{dt}x$$

The last relation can be rewritten as the equation on unknown functions $T_i(\cdot)$

$$Ay + Bu = \frac{\partial}{\partial x}[T(x)] (f(x) + g(x)u) = \begin{bmatrix} \frac{\partial}{\partial x}T_1(x) \\ \vdots \\ \frac{\partial}{\partial x}T_n(x) \end{bmatrix} (f(x) + g(x)u)$$
How to find $T(\cdot)$, $\alpha(\cdot)$ and $\beta(\cdot)$?

The equations on unknown functions $T_i(\cdot)$ are

$$A \begin{bmatrix} T_1(x) \\ \vdots \\ T_n(x) \end{bmatrix} + Bv = \begin{bmatrix} \frac{\partial}{\partial x} T_1(x) \\ \vdots \\ \frac{\partial}{\partial x} T_n(x) \end{bmatrix} (f(x) + g(x)u)$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
How to find $T(\cdot), \alpha(\cdot)$ and $\beta(\cdot)$?

The equations on unknown functions $T_i(\cdot)$ are

$$A \begin{bmatrix} T_1(x) \\ \vdots \\ T_n(x) \end{bmatrix} + B v = \begin{bmatrix} \frac{\partial}{\partial x} T_1(x) \\ \vdots \\ \frac{\partial}{\partial x} T_n(x) \end{bmatrix} (f(x) + g(x) u)$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\mathcal{L}_f T_1 + [\mathcal{L}_g T_1] u = T_2, \quad \mathcal{L}_f T_2 + [\mathcal{L}_g T_2] u = T_3, \quad \ldots$$

$$\mathcal{L}_f T_n + [\mathcal{L}_g T_n] u = v$$
How to find $T(\cdot)$, $\alpha(\cdot)$ and $\beta(\cdot)$?

The equations on unknown functions $T_i(\cdot)$ are

$$A \begin{bmatrix} T_1(x) \\ \vdots \\ T_n(x) \end{bmatrix} + Bv = \begin{bmatrix} \frac{\partial}{\partial x} T_1(x) \\ \vdots \\ \frac{\partial}{\partial x} T_n(x) \end{bmatrix} (f(x) + g(x)u)$$

or the same

$$\mathcal{L}_f T_1 + [\mathcal{L}_g T_1] u = T_2, \quad \mathcal{L}_f T_2 + [\mathcal{L}_g T_2] u = T_3, \quad \ldots$$

$$\mathcal{L}_f T_n + [\mathcal{L}_g T_n] u = v$$

If we assume that $T_1(\cdot), \ldots, T_n(\cdot)$ are independent on $u$, then

$$\mathcal{L}_g T_1 = \mathcal{L}_g T_2 = \cdots = \mathcal{L}_g T_{n-1} = 0, \quad \mathcal{L}_g T_n \neq 0$$
How to find $T(\cdot)$, $\alpha(\cdot)$ and $\beta(\cdot)$?

The equations on unknown functions $T_i(\cdot)$ are

$$
\mathcal{L}_f T_1 + [\mathcal{L}_g T_1] u = T_2, \quad \mathcal{L}_f T_2 + [\mathcal{L}_g T_2] u = T_3, \quad \ldots
$$

$$
\mathcal{L}_f T_n + [\mathcal{L}_g T_n] u = v
$$

If we assume that $T_1(\cdot), \ldots, T_n(\cdot)$ are independent on $u$, then

$$
\mathcal{L}_g T_1 = \mathcal{L}_g T_2 = \cdots = \mathcal{L}_g T_{n-1} = 0, \quad \mathcal{L}_g T_n \neq 0
$$

The equations to solve become

$$
\mathcal{L}_f T_i = T_{i+1}, \quad i \in \{1, \ldots, n - 1\}, \quad \mathcal{L}_f T_n + [\mathcal{L}_g T_n] u = v
$$
How to find $T(\cdot), \alpha(\cdot)$ and $\beta(\cdot)$?

The equations on unknown functions $T_i(\cdot)$ are

$$\mathcal{L}_f T_1 + [\mathcal{L}_g T_1] u = T_2, \quad \mathcal{L}_f T_2 + [\mathcal{L}_g T_2] u = T_3, \quad \ldots$$

$$\mathcal{L}_f T_n + [\mathcal{L}_g T_n] u = v$$

If we assume that $T_1(\cdot), \ldots, T_n(\cdot)$ are independent on $u$, then

$$\mathcal{L}_g T_1 = \mathcal{L}_g T_2 = \cdots = \mathcal{L}_g T_{n-1} = 0, \quad \mathcal{L}_g T_n \neq 0$$

The equations to solve become

$$\mathcal{L}_f T_i = T_{i+1}, \quad i \in \{1, \ldots, n-1\}, \quad \mathcal{L}_f T_n + [\mathcal{L}_g T_n] u = v$$

$$\mathcal{L}_{[f,g]} T_1 = \mathcal{L}_f [\mathcal{L}_g T_1] - \mathcal{L}_g [\mathcal{L}_f T_1] = 0 - \mathcal{L}_g T_2 = 0$$
How to find $T(\cdot)$, $\alpha(\cdot)$ and $\beta(\cdot)$?

The equations on unknown functions $T_i(\cdot)$ are

$$\mathcal{L}_f T_1 + [\mathcal{L}_g T_1] u = T_2, \quad \mathcal{L}_f T_2 + [\mathcal{L}_g T_2] u = T_3, \quad \ldots$$

$$\mathcal{L}_f T_n + [\mathcal{L}_g T_n] u = v$$

If we assume that $T_1(\cdot), \ldots, T_n(\cdot)$ are independent on $u$, then

$$\mathcal{L}_g T_1 = \mathcal{L}_g T_2 = \cdots = \mathcal{L}_g T_{n-1} = 0, \quad \mathcal{L}_g T_n \neq 0$$

The equations to solve become

$$\mathcal{L}_{[f,g]} T_1 = 0, \quad \mathcal{L}_{[f,[f,g]]} T_1 = 0, \quad \ldots, \quad \mathcal{L}_{ad_f^{n-2} g} T_1 = 0, \quad \mathcal{L}_{ad_f^{n-1} g} T_1 \neq 0$$

We need to find only $T_1(\cdot)$, then $T_2(\cdot), \ldots, T_n(\cdot)$ are computed!
How to find $T(\cdot)$, $\alpha(\cdot)$ and $\beta(\cdot)$?

The partial differential equations

$$\mathcal{L}_{X_i} T_1 = 0, \quad i = \{1, \ldots, n - 1\}, \quad \mathcal{L}_{X_n} T_1 \neq 0$$

with

$$X_1 = \text{ad}_f^0 g, \; X_2 = \text{ad}_f^1 g, \ldots, \; X_{n-1} = \text{ad}_f^{(n-2)} g, \; X_n = \text{ad}_f^{(n-1)} g$$

must satisfy properties of the Frobenius Theorem:

The distribution $\triangle$ defined as $\text{span}\{X_1, X_2, \ldots, X_n\}$ should

- be of constant rank, i.e. $X_i$ are to be linear independent
- be involutive

If both conditions are satisfied, then the solution $T_1(x)$ exists!
Lecture 16: Geometric Nonlinear Control

- Background and Useful Concepts
- Feedback Linearization
- Feedback Linearization for Single Link Flexible Joint
The equations of motion are

\[ I \ddot{q}_1 + Mgl \sin(q_1) + k(q_1 - q_2) = 0 \]

\[ J \ddot{q}_2 + k(q_2 - q_1) = u \]
The equations of motion are

\[ \begin{align*}
I \ddot{q}_1 + Mgl \sin(q_1) + k(q_1 - q_2) &= 0 \\
J \ddot{q}_2 + k(q_2 - q_1) &= u
\end{align*} \]

The system is feedback linearizable if \( k \neq 0, \ I \neq 0, \ J \neq 0 \)!