Estimating Dynamics for (DC-motor)+(1st Link) of the Furuta Pendulum

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Abstract

Here the steps done for identification of dynamics for (DC-motor)+(1st Link) of the Furuta Pendulum are described.

I. Problem Formulation

The Furuta pendulum consists of an arm rotating in the horizontal plane and a pendulum attached to the end of the arm moving freely in the vertical plane, see Fig. 1. The equations of motion of the Furuta pendulum, derived from the standard Euler-Lagrange dynamics, are

\[
\begin{align*}
(\alpha + \beta \sin^2 \theta) \ddot{\phi} + \gamma \cos \theta \ddot{\theta} + 2\beta \dot{\theta} \dot{\phi} \sin \theta \cos \theta & - \gamma \dot{\theta}^2 \sin \theta = \tau_{\phi}, \\
\gamma \cos \theta \ddot{\phi} + \beta \ddot{\theta} - \beta \dot{\phi}^2 \sin \theta \cos \theta - \delta \sin \theta & = \tau_{\theta}.
\end{align*}
\]

Here $\phi$ is the angle of the arm; $\theta$ is the angle of the pendulum, measured from the upright position; $\alpha$, $\beta$, $\delta$ and $\gamma$ are constants, defined by various physical parameters of the set-up; $\tau_{\phi}$ and $\tau_{\theta}$ are external torques applied to the arm and to the pendulum, respectively.

The external torque $\tau_{\phi}$ consists of various components, among which we recognize the control input $u$, the friction torque and the contribution due to unmodelled dynamics:

\[
\tau_{\phi} = u - F_{friction}(\cdot) - F_{un.dynamics}(\cdot)
\]

In turn, the control torque $u$ is assumed to be proportional to a voltage applied to the DC-motor

\[
u = K_{DC} \cdot v.
\]

Many of the physical parameters (converted to $\alpha$, $\beta$, $\delta$, $\gamma$ and $K_{DC}$) of the system are unknown, and to be identified. To reduce complexity of the problem, we will consider separately the dynamics of the arm of the Furuta pendulum, where to take into account the mass of the pendulum, we have changed the pendulum to a symmetric metal bob added to the end of the arm, see Fig. 2. With such modification the

\footnote{The dynamics of the DC-motor is assumed to be fast and therefore negligible.}
equation of motion of the arm becomes

\[ \alpha \cdot \ddot{\phi} = K_{DC} \cdot v - F_{friction}(\cdot) - F_{un.dynamics}(\cdot) \]  

(1)

**Problem:** Compute estimates for \( \alpha \), \( K_{DC} \) and suggest a procedure for compensating the friction \( F_{friction}(\cdot) \) in (1).
II. INITIAL STEPS FOR FRICITION COMPENSATION

The friction torque $F_{friction}(\cdot)$ in (1) is a nonlinear unknown mapping acting in opposite direction to the angular velocity of the arm, $\frac{d\phi}{dt}$. It is often the case that the nature and the precise form of that mapping is not needed, but its approximation that can be used for (partly) compensating friction, is of interest. For example, stiction levels for $F_{friction}(\cdot)$ can be readily identified from the equation (1). Indeed, if one applies ramp signals (with a small inclination) for $v$, then the levels of control signal when the arm starts moving can be used for such purpose. Here it worth mentioning that the value of $K_{DC}$ might be unknown provided that one is interested to compensate the friction by adding a feedforward term.

Such experiments for Furuta pendulum reveal the following control signal levels that can be used to compensate a Coulomb part of the friction torque:

$$v_{Coulomb} = \begin{cases} 
0.032, & \text{if } \frac{d\phi}{dt} > 0 \\
-0.033, & \text{if } \frac{d\phi}{dt} < 0 
\end{cases}$$

(2)

In fact, these values are varying a bit being functions of the arm angle $\phi$. 
Obviously, the friction torque $F_{\text{friction}}(\cdot)$ in (1) has other components, for instance, a viscous part. It will be identified later, while adding to the control the feedforward signal $v_{\text{Coulomb}}$.

$$v = v_{\text{nominal}} + v_{\text{Coulomb}},$$

removes strong nonlinearity in the system making the dynamics of (1) close to linear

$$\alpha \cdot \ddot{\phi}(t) = K_{\text{DC}} \cdot \left( v_{\text{nominal}}(t) + v_{\text{Coulomb}}(t) \right) - F_{\text{friction}}(\cdot) - F_{\text{un.dynamics}}(\cdot)$$

Here $[\beta \dot{\phi}(t)]$ is the viscous part of the friction and $e(t)$ is the remaining terms interpreted as a noise.

III. ORGANIZATION OF EXPERIMENTS AND DATA PREPROCESSING

It is expected that the system with the feedforward compensation term (2) should behave as

$$\alpha \cdot \ddot{\phi}(t) + \beta \cdot \dot{\phi}(t) = K_{\text{DC}} \cdot v_{\text{nominal}}(t) + e(t).$$ (3)

The parameters $\alpha$, $\beta$, $K_{\text{DC}}$ are to be identified from the measured values

$$y(t) = \phi(t) + e_m(t)$$ (4)

Here $e_m(t)$ is the measurement noise. In addition, we have freedom in choosing the nominal control signal $v_{\text{nominal}}(t)$. The system (3) is unstable (the arm moves in horizontal plane and its potential energy is constant). It is suggested

- to introduce a feedback action to stabilize the system;
- to use a reference signal to the closed loop signal of a particular pass-band characteristic.

The simplest choice of a stabilizing controller for (3) is the proportional feedback

$$v_{\text{nominal}}(t) = K_p \cdot \left( r(t) - y(t) \right) = K_p \cdot \left( r(t) - \{ \phi(t) + e_m(t) \} \right)$$

where $r(t)$ is the reference. With such choice the system (3) becomes

$$\alpha \cdot \ddot{\phi}(t) + \beta \cdot \dot{\phi}(t) + K_{\text{DC}} \cdot K_p \cdot \phi(t) = K_{\text{DC}} \cdot K_p \cdot r(t) + \left\{ e(t) + K_p \cdot e_m(t) \right\}.$$ (5)

It is stable provided that $K_p > 0$. We have performed 3 experiments, where the feedback gain $K_p$ was

$$K_p = 0.05, \quad K_p = 0.1, \quad K_p = 0.15,$$ (6)

they are all positive constants
and the reference $r(t)$ were built of sinusoids from the frequency range $[2, 6]$ rad/sec. The zero frequency signal has been intentionally excluded from that interval, because the Coulomb friction (2) is not well defined at zero angular velocity\(^3\). The recorded data and magnitudes of their fast Fourier transforms are depicted on Figs. 3-5.

The sampling period for the recorded data was $T_s = 0.0002$ [sec] (the corresponding Nyquist frequency $\frac{\pi}{T} \approx 15750$ [rad/sec]), the data have been re-sampled to get Nyquist frequency about 12-13 [rad/sec] (that is about 2.5-3 times larger than the bandwidth of the power spectrum of the recorded signals). The re-sampled signals for the 3rd experiment (with the sampling period $T_{rs} = 0.252$ [sec]) are shown on Fig.6. The data for all three experiments have been further processed to remove the mean values and possible ramps prior re-sampling.

IV. NON-PARAMETRIC METHODS

A. Impulse and Step Responses

Computing the estimates for step and impulse responses of the system are the first steps to do. They can show us various properties including a time constant of the system, possible delay, presence of feedback in the data etc. Both step and impulse responses can be computed for the re-sampled and original experimental data.

It turns out that computing step and impulse responses for the original data leads to results, which are difficult to use for some reasoning (except the fact that the system is not a low-pass filter), see Figs 7 and 8. This is indeed expectable, the system has been excited to show oscillatory behavior in response (a resonance behavior). It implies that the settling time for the system’s step response could be quite large (3-4 seconds). With the sampling period $T_s = 0.0002$ [sec] one needs to estimate more than 15-20 thousands of coefficients to get corresponding behavior. Such task is too demanding.

The computing the step and impulse responses for the re-sampled data is more feasible numerically, but the estimates will be of large variance, and again become difficult to interpret.

B. Spectral Analysis

Computing estimates for frequency response of the system and for spectral density of noise, can be done via relating fast Fourier transforms of the input $r(t)$ and the output $y(t)$. Here the re-sampled data are used for focusing on important range of frequencies. The estimates are dependent on various parameters including a type and size of lag window chosen for computing spectra, see Figs. 9 and 10.

\(^3\)When the angular velocity of the arm crossing zero level, the friction map has to have memory state in its description. The friction model (2) is the memoryless mapping.
The phase plots of estimates, see Figs. 9 and 10, show that the systems are likely of second order (the phase changes from 0 to $-\pi$), while the magnitude plots show that one can expect resonances at $\omega = 2.5 - 3.0$ and 4.5 [rad/sec] for the first and the second closed systems (5), (6).

V. PARAMETRIC METHOD

Both the model (5) of the system obtained from basic principles (Newton laws) and the spectral analysis of the data suggest that the order of the model equals to 2, while delay in response is not substantial. One can then use the prediction error method to recover estimates of parameters for the second order system

$$\hat{G}(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$$

(7)

making various modification of focus in estimating and transforming between continuous and discrete time models structures. Comparing the model (5) with the transfer function (7), it is expected that

$$b_1 \approx 0, \quad b_0 \approx a_0, \quad a_1 \approx \frac{\beta}{\alpha}, \quad a_0 \approx \frac{K_{DC} K_p}{\alpha}$$

(8)

Prior to start searching for parameters, one should take into account that

- the spectra of data is within interval $[2, 6]$ [rad/sec], therefore an estimate for $b_0$ of system (5) will be unreliable and of a large variance;

- the estimation procedure might result into an unstable system. Indeed, the viscous friction in the system is small, hence one should expect two resonance poles close to imaginary axis, and they can be identified with mistake bringing estimated poles of the system into the region of instability. The experiments show that the system (5), (6) is stable, then an appropriate projection ensuring stability for an estimated system should be implemented.

As seen from the accompanying MATLAB file, the first two guesses in (8) are wrong, the value for $b_1$ differs from zero and $b_0$ is almost 10 times larger than $a_0$ for all the estimated models and for all experiments. This mismatch should be explained.

At the same for all three experiments the values of $a_0/K_p = K_{DC}/\alpha$ become close to each other

Experiment 1: $$\frac{a_0}{K_p} = 121.37, \quad \sigma = 1.15$$

Experiment 2: $$\frac{a_0}{K_p} = 126.16, \quad \sigma = 0.3467$$

Experiment 3: $$\frac{a_0}{K_p} = 124.69, \quad \sigma = 0.2358$$

These mean values and their variances are used for estimating $K_{DC}/\alpha$ value:

$$\frac{K_{DC}}{\alpha} = 124.8629, \quad \sigma = 0.1251.$$ 

VI. VALIDATION

The estimated models were successfully tested on the part of the recorded data left for validation step.
RECORDED DATA (EXPERIMENT 1)

Fig. 3. (a) The recorded responses of the arm of the Furuta Pendulum under the proportional feedback (5) with $K_p = 0.05$; (b) the magnitudes of FFT of time signals
Fig. 4. (a) The recorded responses of the arm of the Furuta Pendulum under the proportional feedback (5) with $K_p = 0.1$; (b) The magnitudes of FFT of time signals
Fig. 5. (a) The recorded responses of the arm of the Furuta Pendulum under the proportional feedback (5) with $K_p = 0.15$; (b) The magnitudes of FFT of time signals
RECORDED vs. RESAMPLED DATA (EXPERIMENT 3)

Fig. 6. The time evolution of the data from the experiment 3 (sampled with $T_s = 0.0002$ [sec]) are depicted together with the re-sampled copy when $T_{rs} = 0.252$ [sec] for the time interval between 8 and 38 [sec].

the estimated impulse response for resampled data from EXPERIMENT N2

Fig. 7. The impulse response estimate for the original data from experiment 2.
Fig. 8. The step response estimate for the original data from experiment 2.

Fig. 9. The spectral estimate for frequency response for experiment 1 with the re-sampled data ($T_{rs} = 0.252 \text{ [sec]}$) and Hamming window used in computing with the sizes 15, 25 and 50.
Fig. 10. The spectral estimate for frequency response for experiment 2 with the re-sampled data ($T_{rs} = 0.252$ [sec]) and Hamming window used in computing with the sizes 15, 25 and 50.