CHAPTER 11

Dynamic Behavior of Measurement Systems

The general concept of the dynamic response of measuring systems was introduced in Chapter 2. In this chapter the mathematical basis for measuring dynamic response is presented and applied to important sensors and transducers.

11.1 ORDER OF A DYNAMIC MEASUREMENT SYSTEM

For static measurements, the mathematical relationship between the measurand (input) and the measurement-system output can be described with algebraic equations. For example, for a linear device, the input–output relationship can be given by

\[ y = Kx \]  \hspace{1cm} (11.1)

where \( y \) is the measurement-system output, \( x \) is the value of the measurand, and \( K \) is the instrument static sensitivity. For time-varying measurements, however, the mathematical relationship between the measurand and the output must be described with a differential equation. The equation describing the behavior of an instrument with a time-varying input is derived by applying the appropriate basic physical principles (such as conservation of energy, Newton’s second law of motion, or Ohm’s law) to the instrument as it interacts with its environment. In general, the input–output relationship of a linear measurement system can be expressed in the form of an ordinary differential equation (ODE):

\[ a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = bx \]  \hspace{1cm} (11.2)

In this equation, \( n \) is the order of the system, \( x \) is the input signal (forcing function), \( y \) is the output signal, and \( a_0 \cdots a_n \) are constant coefficients that depend on the characteristics of the measurement system. \( x \) and \( y \) are, in general, both functions of time, \( x(t) \) and \( y(t) \). Most measuring devices can be described using Eq. (11.2) with \( n \) values of 0, 1, or 2.
In the following sections, some important dynamic characteristics of measurement systems are discussed. A more detailed presentation is given by Doebelin (1990).

11.2 ZERO-ORDER MEASUREMENT SYSTEMS

If \( n = 0 \), Eq. (11.2) reduces to

\[
a_0 y = bx
\]  

(11.3)

Although \( x \) varies dynamically, \( y \) responds proportionally and the measurement process is essentially a static measurement. This is the ideal measurement process. No system is truly of order zero, but many systems approximate this behavior in some modes of their operation.

Consider the simple displacement potentiometer shown in Figure 11.1. In this system a fixed voltage \( E \) is applied to a resistor with length \( L \). The instrument is used to measure displacement and generate an output voltage \( e \), which is proportional to the position, \( x \), of the slider. The output can be related to the input by applying Ohm’s law to give

\[
e = \frac{E}{L} x
\]  

(11.4)

For a wide range of operation, the output \( e \) is unaffected by the nature of the time variation of \( x \) and shows zero-order behavior. However, if \( x \) varies extremely rapidly, inductive, capacitance, and flexural effects may invalidate Eq. (11.4), and the system can no longer be described as zero order. Strain-gage sensors can be considered as zero-order systems for most of their range of operation. However, many transducers using strain-gage sensors, such as pressure transducers, are in fact second-order systems.

11.3 FIRST-ORDER MEASUREMENT SYSTEMS

First-order systems contain only a single mode of energy storage. The simple \( R-C \) circuit is a first-order electrical system. The most common first-order measurement systems are thermal systems such as temperature-measuring devices, which include thermal capacitance and resistance to heat flow. These show first-order behavior over most ranges of operation. First-order analysis can also be used for some operational modes of mechanical systems that include friction and springs, but in which the inertial effects of mass can be neglected. (Certain applications of pressure transducers are examples.)

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**FIGURE 11.1**

Linear potentiometer, zero-order system.
11.3.1 Basic Equations

For \( n = 1 \), Eq. (11.2) becomes

\[
\frac{dy}{dt} + a_0y = bx \tag{11.5}
\]

This equation can be rewritten in the form

\[
\frac{dy}{dt} + y = Ky \tag{11.6}
\]

\( K \) is the static sensitivity, and \( \tau \), which has units of time, is called the time constant.\(^1\) The time constant, \( \tau \), is determined by the physical characteristics of the measuring system. If the time-derivative term were to be neglected, the system would show static behavior and \( y \) would have the ideal response value, \( Kx \). However, the existence of the time-derivative term introduces a dynamic measurement error.

The solution of Eq. (11.6) is the sum of two parts. One part is called the general solution and is determined by the system itself and is independent of the forcing function \( x \). The general solution is obtained by solving Eq. (11.6) assuming that the right-hand side, \( Kx \), is zero. The other part of the solution, called the particular solution, is dependent on the input forcing function, \( x \), and is a solution to the complete Eq. (11.6). In the following sections, the time responses of first-order systems to step, ramp, and sinusoidal inputs are discussed.

11.3.2 Step Input

Consider a situation in which both the input and output to the device are both zero until time \( t = 0 \). At \( t = 0 \), as shown in Figure 11.2(a), the input \( x \) suddenly rises to a value \( x_0 \). If the device were ideal (zero order), it would produce the output \( y_r = Kx_0 \), which is independent of time. For the actual first-order device, we seek the time variation of the device output, \( y \). The general solution to Eq. (11.6) is

\[
y_{\text{gen}} = Ce^{-\frac{t}{\tau}} \tag{11.7}
\]

where \( C \) is a constant to be determined. The particular solution is

\[
y_{\text{part}} = Kx_0 \tag{11.8}
\]

The complete solution to this problem is the sum of the general and particular solutions and is given by

\[
y = Kx_0(1 - e^{-\frac{t}{\tau}}) \tag{11.9a}
\]

\(^1\)In evaluating an instrument, the static sensitivity will have units of the physical output divided by the units of the input. For example, a thermocouple output is in millivolts, and the instrument sensitivity will be in units such as mV/°C. When the instrument is used, however, the instrument output will be converted to measurement units with a calibration function. Thus, as used, the thermocouple will have a sensitivity with units of (output degrees)/(input degrees) and the static sensitivity will effectively be unity. Dynamic measurement errors may be evaluated in either the physical output units of the instrument or in the units after the calibration function has been applied.
where the constant $C$ has been eliminated using the initial condition that $y = 0$ when $t = 0$. The validity of Eq. (11.9a) can be demonstrated by substituting it into Eq. (11.6) and noting that it also satisfies the initial condition.

Figure 11.2(b) shows the variation of the output $y$ with respect to the time $t$. At $t = \tau$ the output has reached 63.3% of its final value, and at $t = 4\tau$ it has reached 98.2% of its final value. Since the larger the time constant of a first-order system, the longer it will take the output to approach its final value, the system should have a small time constant if rapid response is desired.

In many cases, the output does not have an initial value of zero. In this case, Eq. (11.9a) can be viewed as showing the change in output for a change in input. If the initial value of $y$ is $y_i$ and the equilibrium value (a long time after the step change) is $y_e (y_e = Kx_0)$, then Eq. (11.9a) will take the form

$$\frac{y - y_i}{y_e - y_i} = (1 - e^{-t/\tau})$$

(11.9b)

This is the form usually used to evaluate measurement devices.

### 11.3.3 Ramp Input

In this case both $y$ and $x$ are zero at time $t = 0$. As shown in Figure 11.3(a), starting at $t = 0$, the forcing function $x$ takes the form

$$x = At$$

(11.10)
where $A$ is the slope of the ramp. The ideal, zero-order response would then be $y_e = KA\tau$. As in the case of the step input, Eq. (11.6) is the applicable differential equation and Eq. (11.7) is the general solution (which is independent of the forcing function). The particular solution for the case of the ramp input is

$$y_{\text{part}} = KA(t - \tau) \quad (11.11)$$

The solution is the sum of general and particular solutions, and evaluating $C$ using the initial condition that $y = 0$ when $t = 0$, we obtain the solution

$$y = KA(\tau e^{-\frac{t}{\tau}} + t - \tau) \quad (11.12a)$$

Again, this result can be verified by substituting it into the differential equation and noting that it satisfies the initial condition. Figure 11.3(b), a sketch of Eq. (11.12a), shows that there will be an error in the output with respect to the input. After a few time constants have elapsed, the error is the difference between the output value and the ideal output value one time constant earlier. Thus the error will be minimized by using devices with small values of the time constant.

In most cases, $y$ will not be zero at time zero. If $y_i$ is the initial value of $y$ at $t = 0$, then Eq. (11.12a) can be expressed as

$$y = y_i + KA(\tau e^{-\frac{t}{\tau}} + t - \tau) \quad (11.12b)$$

This form is more general and is the form typically used.
11.3.4 Sinusoidal Input

The third input function considered for the first-order system is the continuing sinusoidal input. In this case we take the forcing function to have the form

\[ x = x_0 \sin \omega t \]  

(11.13)

For an ideal, zero-order system the output would be \( y_e = Kx_0 \sin \omega t \). As with the step and ramp inputs, the solution can be viewed as the sum of the general and particular solutions. However, in the case of sinusoidal input, it is the continuing response that is of greatest interest, and the transient caused by the initial application of the forcing function can be neglected. The continuing response corresponds to the particular solution, which is given by

\[ y = \frac{Kx_0}{\sqrt{1 + \omega^2 \tau^2}} \sin(\omega t + \phi) \]  

(11.14)

where \( \phi \) is the phase angle between the forcing function and the output response. \( \phi \) is given by

\[ \phi = -\tan^{-1} \frac{\omega \tau}{1} \]  

(11.15)

Figure 11.4(a) presents the ratio of the actual amplitude of \( y \) to the ideal amplitude, \( Kx_0 \), versus the nondimensionalized angular frequency, \( \omega \tau \). For the continuing response given, the initial input and output conditions do not matter. Figure 11.4(b) presents the phase angle. These figures show that if the product of the time constant and the forcing-function angular frequency approaches zero, the amplitude of the output will approach the ideal case and the phase angle will approach zero. Otherwise, the output magnitude and phase will have a systematic error with respect to the actual input.

This analysis considers the response to a sinusoidal input of a single frequency. As discussed in Chapter 5, periodic inputs of any general form can be broken down into a series of single-frequency sine and cosine functions using Fourier analysis. The response of the system to these individual Fourier components can then be analyzed using the previous results. The foregoing discussion applies to the dynamic behavior of any system that can be modeled by a time-dependent first-order ordinary differential equation. Further details on the solution can be found in Doebelin (1990).

11.3.5 Thermocouple as a First-Order System

Thermocouples are a very common first-order system and serve to demonstrate the application of first-order dynamic analysis. As shown in Section 2.3, the time constant for a thermometer is given by

\[ \tau = \frac{mc}{hA} \]  

(11.16)

This formula is also applicable to other temperature-measuring devices, such as thermocouples and resistance temperature detectors. To obtain a small time constant, it is necessary to have a high surface area relative to the mass. Since temperature sensors usually have relatively simple geometry (usually, cylinders or spheres), this means that
the sensor should be very small. It is also helpful that the heat transfer coefficient \( h \) increases as the size decreases. In Chapter 9 it was shown that for thermocouples, small size is also desirable to minimize conduction and radiation errors. Unfortunately, small temperature sensors are vulnerable to damage from impact, vibration, and corrosion.

Thermocouples, described in Chapter 9, have a wide range of applications in industry and research. For fast, dynamic response and in research projects, it is best to use thermocouples with bare junctions similar to that shown in Figure 11.5(a). Such bare-junction thermocouples are very vulnerable to damage and corrosion and also present installation problems. As a result, it is common to use thermocouples encased in a metal sheath for mechanical protection [Figure 11.5(b)]. The additional mass of the metal sheath and the insulation increases the heat storage capacity of the instrument, increases its time constant, and consequently, slows its response. In some industrial applications the shielded thermocouple is located in a fluid-filled well for further durability and mechanical and chemical protection [Figure 11.5(c)], further slowing its response. Such thermocouples might have time constants on the order of minutes. Further details on the transient response of thermocouples and methods for determining and reducing thermocouple time constants are presented in Benedict (1984). The following examples show some typical applications of first-order analysis applied to thermocouples.
Example 11.1

A Pt-Pt/13%Rd (type R) bare-junction thermocouple has an approximately spherical junction with a diameter of 0.3 mm. It is used to measure the temperature of gases in a combustion tunnel. When the flame is ignited, it produces an approximate step increase of the gas temperature of 900 K. The average heat-transfer coefficient\(^1\) on the surface of the thermocouple is 500 W/m\(^2\)K. The gas temperature before ignition is 300 K.

(a) Find the time constant of the thermocouple.
(b) After how much time and how many time constants will the measurement error be less than 1% of the final temperature change?
(c) If the same thermocouple is used in an aqueous environment in which the heat transfer coefficient is 6000 W/m\(^2\)K, what will be the thermocouple time constant?
(d) Plot the thermocouple response versus time for both cases.

Solution: For a step change in input temperature, this first-order system is described by Eq. (11.9). We will take the junction properties to be those of platinum. From Table B.5, the properties of platinum are \( \rho = 21,450 \) kg/m\(^3\) and \( c = 134 \) J/kg.K.

\(^1\)To calculate the time constant, we need to calculate the heat transfer coefficient. Holman (2002) presents a comprehensive set of theoretical and empirical formulas that can be used in such applications. For flow over spheres, the nondimensionalized heat transfer coefficient can be expressed in terms of the Reynolds number (\( Re = U \cdot D / \mu \)) as

\[
\frac{hD}{k} = 2 + (0.4 Re^{0.5} + 0.06 Re^{0.7}) \frac{Pr \rho \mu_s}{\mu_w} \left( \frac{\mu_w}{\mu_s} \right)^{1/4}
\]

In this equation, \( h \) is the heat-transfer coefficient, \( D \) is the sphere diameter, and \( \rho, k, \mu_s \), and \( Pr \) are, respectively, the density, thermal conductivity, dynamic viscosity, and Prandtl number of the fluid. Fluid properties (except the two viscosities shown) are evaluated at the fluid temperature, which is the average temperature of the fluid (\( \infty \)) and heat-transfer surface (\( \omega \)).
(a) From Eq. (11.16) the time constant will be

\[ \tau = \frac{mc}{kA} = \frac{\rho \bar{v}_s}{kA} = \left[ 21.450 \times \left( \frac{\pi \times 0.0003}{6} \right) \right] \times \frac{134}{500 \times \pi \times 0.0003} = 0.287 \text{ s} \]

(b) We seek the time when the left side of Eq. (11.9b) has a value of 0.99 (error of 0.01):

\[ \frac{T - T_i}{T_f - T_i} = 0.99 = 1 - e^{-\tau t} \]

Solving, we obtain \( t = 1.32 \text{ s} \), which corresponds to 4.6 time constants.

(c) For the aqueous environment, the time constant will be

\[ \tau = \frac{mc}{kA} = \frac{\rho \bar{v}_s}{kA} = \left[ 21.450 \times \left( \frac{\pi \times 0.0003}{6} \right) \right] \times \frac{134}{6000 \times \pi \times 0.0003} = 0.024 \text{ s} \]

Due to the much higher heat-transfer coefficient in the liquid phase compared with the gas phase, this time constant is an order of magnitude smaller than the time constant for the gas phase.

(d) The actual input and the time response of the thermocouple are shown in Figure E11.1.

*Comment:* Certain assumptions were made in this analysis. It was assumed that the shape of the junction was spherical and that the radiation and conduction effects of connecting wires could be neglected.

**Example 11.2**

In a special test for determining the performance of a heat exchanger, an electric heater produces a temperature variation in an airflow that can be approximated with a 1-Hz sine wave. The air
temperature is measured downstream of the heater with a thermocouple that has a time constant of 0.2 s. The maximum and minimum temperature values measured by the thermocouple are 120° and 100°C, respectively.

(a) Determine the equation for the measured variation of temperature.
(b) Determine the actual variation in the air temperature.
(c) Draw the actual and the indicated temperature as a function of time.

Solution:

(a) To determine the equation for the measured temperature variation, we need to determine the amplitude of the oscillation and the mean temperature:

\[
A = \frac{T_{\text{max}} - T_{\text{min}}}{2} = \frac{120 - 100}{2} = 10\,^\circ\text{C}
\]

\[
T_{\text{av}} = \frac{T_{\text{max}} + T_{\text{min}}}{2} = \frac{120 + 100}{2} = 110\,^\circ\text{C}
\]

The time variation of the indicated temperature is then

\[T = T_{\text{av}} + A \sin 2\pi f t \quad \text{or} \quad T = 110 + 10 \sin 2\pi f t \, ^\circ\text{C}
\]

(b) The amplitude response and the phase response of the thermocouple are determined through Eqs. (11.14) and (11.15). Both the input and output of this measuring system are measured in units of temperature, and consequently, the ideal output amplitude, \( y_r \), would have the same value as the input amplitude, \( x_0 \). Hence, the static sensitivity, \( K \), equals 1. In this case the amplitude of the output, \( y \), is \( A \), and we seek \( x_0 \), the input amplitude. Using Eq. (11.14), the output amplitude can be obtained in terms of the input amplitude:

\[
A = \frac{K x_0}{(1 + \omega^2 \tau^2)^{1/2}}
\]

\[10 = \frac{1 x_0}{[1 + (2\pi \times 1)^2 \times 0.2^2]^{1/2}}
\]

Solving gives

\[x_0 = 16.1\,^\circ\text{C}
\]

Using Eq. (11.15), the phase angle of the measured temperature with respect to the actual temperature will be

\[\phi = -\tan^{-1} \omega \tau = -\tan^{-1}(2\pi \times 1 \times 0.2) = -51.5^\circ = -0.899 \text{ rad}
\]

The actual variation in air temperature is then given by

\[T_{\text{act}} = 110 + 16.1 \sin(2\pi t + 0.899)
\]

(c) The actual and indicated temperature variations are shown in Figure E11.2.
PROBLEMS

11.1 If the thermocouple of Example 11.1 is used to measure the temperature of air that increases at a rate of 10°C/s,
(a) determine the time after which the thermocouple will follow the gas temperature steadily (slope is within 1% of the ideal value).
(b) calculate the continuing error in the temperature measurement after this time.

11.2 Determine the time that temperature measuring devices (such as a thermocouple) will reach 98% of their final value when exposed to a step change in temperature if they have time constants of 0.1, 1, and 10 s.

11.3 If the temperature-measuring devices of Problem 11.2 are exposed to a fluid with sinusoidal temperature variation of 0.1 Hz, determine the ratio of the response amplitude to the input amplitude and the phase lag for each case.

11.4 Which of the devices of Problem 11.2 will you recommend for measuring the temperature in an air-conditioning control system of a building that has a temperature variation of 0.5°C/min?

11.5 A commercial thermocouple that has an effective spherical junction diameter of 2 mm is intended to be used for some transient measurements. You may assume that the thermocouple material has properties similar to those of copper. Estimate the time constants of this thermocouple for gaseous and liquid environments that have convective heat-transfer coefficients of 100 W/m²·°C and 3000 W/m²·°C, respectively.

11.6 A chromel–alumel thermocouple junction, which can be approximated with a sphere, has an effective diameter of 1 mm. It is used to measure the temperature of a gas flow with an effective heat-transfer coefficient of 500 W/m²·°C.
(a) Determine the time constant of this thermocouple.
(b) If the gas temperature suddenly increases by 100°C, how long will it take the thermocouple to attain a temperature rise within 1% of the gas temperature rise?
(c) Find the answer to the previous questions if the diameter of the bead is doubled. (Assume that the heat-transfer coefficient remains the same.)

11.7 The following set of data are produced by a temperature-measuring device (first order) that is suddenly immersed into a mixture of ice and liquid water. Initially, the device is in ambient air (20°C). Determine the time constant of this temperature-measuring device.

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature (°C)</td>
<td>16.7</td>
<td>8.1</td>
<td>3.3</td>
<td>0.6</td>
<td>0.1</td>
</tr>
</tbody>
</table>

11.8 Two copper–constantan thermocouples with 1- and 2-mm diameters are dipped into boiling water from ambient (20°C). The heat-transfer coefficient to the thermocouples is 3000 W/m²·°C. Determine the output from each thermocouple in 1, 5, and 10 s. Estimate the time beyond which they will read the same temperature (within 0.1°C). Assume the properties of copper–constantan to be the same as those of copper, and assume that the junction is spherical.

11.9 A thermometer with a time constant of 10 s is used to measure the temperature of a fluid that is heated by steam. The temperature rise can be approximated with a ramp of 10°C/min. Determine the time delay in the measurement indicated and the instantaneous error in the temperature reading.

11.10 Air is heated by an electric heater. The air temperature output (in °C), as measured by a thermocouple, can be approximated with

\[ T = 200 + 10 \cos 0.5t \]

The time constant of the thermocouple is approximately 5 s.
(a) Determine the average, maximum, and minimum measured temperatures.
(b) Estimate the actual air temperature as a function of time.
CHAPTER 11

11.1 (a) 1.32 sec (b) $\Delta T = 2.86^\circ C$

11.3 $\tau = 0.1, y/ye = 0.998, \phi = 0.063$ rad, $\tau = 1, y/ye = 0.847, \phi = 0.56$ rad,
    $\tau = 10, y/ye = 0.157, \phi = 1.41$ rad

11.5 $\tau_{gas} = 11.45$ sec, $\tau_{eq} = 0.38$ sec

11.7 $\tau = 0.55$ sec

11.9 10 sec, 1.67$^\circ$C

11.11 $\omega_n = 14.1$ rad/sec, $\zeta = 0.007$

11.15 $y/Kx_0 = 1.05, \phi = -2.7^\circ$

11.17 982.4 rad/sec, 54 msec

11.20 1 kHz

11.26 $f_n = 626$ Hz, $\zeta = 1.6 \times 10^{-6}$

11.28 154 Hz