Lecture 10: Dynamics: Euler-Lagrange Equations

• Examples
Lecture 10: Dynamics: Euler-Lagrange Equations

- Examples
- Holonomic Constraints and Virtual Work
The second Newton law says that the equation of motion of the particle is

\[ m \frac{d^2 y}{dt^2} = \sum_i F_i = f - mg \]

- \( f \) is an external force;
- \( mg \) is the force acting on the particle due to gravity.
Example

The equation of motion of the particle is

\[ m \frac{d^2 y}{dt^2} = \sum_i F_i = f - mg \]

can be rewritten in the different way!
Example

The equation of motion of the particle is

\[ m \frac{d^2 y}{dt^2} = \sum_i F_i = f - mg \]

can be rewritten in the different way!

Some parts of the equation of motion is equal to

\[ m \frac{d^2 y}{dt^2} = \frac{d}{dt} \left( m \frac{d y}{dt} \right) = \frac{d}{dt} \left( m \frac{\partial}{\partial \dot{y}} \left[ \frac{1}{2} \dot{y}^2 \right] \right) = \frac{d}{dt} \left( \frac{\partial}{\partial \dot{y}} \mathcal{K} \right) \]
Example

The equation of motion of the particle is

\[ m \frac{d^2 y}{dt^2} = \sum_i F_i = f - mg \]

can be rewritten in the different way!

Some parts of the equation of motion is equal to

\[
\begin{align*}
    m \frac{d^2 y}{dt^2} &= \frac{d}{dt} \left( m \frac{dy}{dt} \right) = \frac{d}{dt} \left( m \frac{\partial}{\partial \dot{y}} \left[ \frac{1}{2} \dot{y}^2 \right] \right) = \frac{d}{dt} \left( \frac{\partial}{\partial \dot{y}} K \right) \\
    mg &= \frac{\partial}{\partial y} \left[ mgy \right] = \frac{\partial}{\partial y} P
\end{align*}
\]

with kinetic/potential energies defined by \( K = \frac{1}{2} m \dot{y}^2 \), \( P = mgy \)
Example

The equation of motion of the particle is

\[ m \frac{d^2 y}{dt^2} = \sum_i F_i = f - mg \]

can be rewritten in the different way!

Some parts of the equation of motion is equal to

\[ m \frac{d^2 y}{dt^2} = \frac{d}{dt} \left( m \frac{dy}{dt} \right) = \frac{d}{dt} \left( m \frac{\partial}{\partial \dot{y}} \left[ \frac{1}{2} \dot{y}^2 \right] \right) = \frac{d}{dt} \left( \frac{\partial}{\partial \dot{y}} K \right) \]

\[ mg = \frac{\partial}{\partial y} \left[ mgy \right] = \frac{\partial}{\partial y} P \]

with kinetic/potential energies defined by \( K = \frac{1}{2} m \dot{y}^2, \ P = mgy \)

Then the second Newton law can be rewritten as

\[ \frac{d}{dt} \left( \frac{\partial}{\partial \dot{y}} \mathcal{L} \right) - \frac{\partial}{\partial y} \mathcal{L} = f \quad \text{with} \quad \mathcal{L} = K - P \]

where the function \( \mathcal{L}(y, \dot{y}) \) is called the Lagrangian.
Example:

A rigid link ($\theta_l$) coupled through a gear to DC motor ($\theta_m = r\theta_l$):
Example:

A rigid link ($\theta_l$) coupled through a gear to DC motor ($\theta_m = r\theta_l$):

- Kinetic energy: $\mathcal{K} = \frac{1}{2} J_m \dot{\theta}_m^2 + \frac{1}{2} J_l \dot{\theta}_l^2 = \frac{1}{2} (r^2 J_m + J_l) \dot{\theta}_l^2$
Example:

A rigid link ($\theta_l$) coupled through a gear to DC motor ($\theta_m = r\theta_l$):

- Kinetic energy: $\mathcal{K} = \frac{1}{2} J_m \dot{\theta}_m^2 + \frac{1}{2} J_l \dot{\theta}_l^2 = \frac{1}{2} (r^2 J_m + J_l) \dot{\theta}_l^2$
- Potential energy: $\mathcal{P} = Mgl (1 - \cos \theta_l)$
Example:

A rigid link ($\theta_l$) coupled through a gear to DC motor ($\theta_m = r \theta_l$):

- Kinetic energy: $\mathcal{K} = \frac{1}{2} J_m \dot{\theta}_m^2 + \frac{1}{2} J_l \dot{\theta}_l^2 = \frac{1}{2} (r^2 J_m + J_l) \dot{\theta}_l^2$
- Potential energy: $\mathcal{P} = Mgl \left(1 - \cos \theta_l \right)$
- the Lagrangian is $\mathcal{L} = \mathcal{K} - \mathcal{P}$ and the dynamics are

$$\frac{d}{dt} \left( \frac{\partial}{\partial \dot{\theta}_l} \mathcal{L} \right) - \frac{\partial}{\partial \theta_l} \mathcal{L} = ru$$
Example:

A rigid link \((\theta_l)\) coupled through a gear to DC motor \((\theta_m = r\theta_l)\):

- Kinetic energy: \(K = \frac{1}{2} J_m \dot{\theta}_m^2 + \frac{1}{2} J_l \dot{\theta}_l^2 = \frac{1}{2} (r^2 J_m + J_l) \dot{\theta}_l^2\)
- Potential energy: \(P = Mgl (1 - \cos \theta_l)\)
- the Lagrangian is \(\mathcal{L} = K - P\) and the dynamics are
  \[(r^2 J_m + J_l) \ddot{\theta}_l + Mgl \sin \theta_l = ru\]
Lecture 10: Dynamics: Euler-Lagrange Equations

- Examples
- Holonomic Constraints and Virtual Work
Concept of Holonomic Constraint

Unconstrained system of $k$ particles has $3k$ degrees of freedom. The number of Dof is less if the particles are constrained.
Concept of Holonomic Constraint

A constraint imposed on \( k \) particles (with coordinates \( r_1, r_2, \ldots, r_k \in \mathbb{R}^3 \)) is called holonomic, if it is of the form

\[
g_i(r_1, r_2, \ldots, r_k) = 0, \quad i = 1, 2, \ldots, l
\]
Concept of Holonomic Constraint

A constraint imposed on \( k \) particles (with coordinates \( r_1, r_2, \ldots, r_k \in \mathbb{R}^3 \)) is called \textit{holonomic}, if it is of the form

\[
g_i(r_1, r_2, \ldots, r_k) = 0, \quad i = 1, 2, \ldots, l
\]

For example, given two particles joined by massless rigid wire of length \( l \), then

\[
r_1, r_2 \in \mathbb{R}^3 : \| r_1 - r_2 \|^2 = (r_1 - r_2)^T (r_1 - r_2) = l^2
\]
Concept of Holonomic Constraint

A constraint imposed on \( k \) particles (with coordinates \( r_1, r_2, \ldots, r_k \in \mathbb{R}^3 \)) is called holonomic, if it is of the form

\[
g_i(r_1, r_2, \ldots, r_k) = 0, \quad i = 1, 2, \ldots, l
\]

For example, given two particles joined by massless rigid wire of length \( l \), then

\[
r_1, r_2 \in \mathbb{R}^3 : \quad \|r_1 - r_2\|^2 = (r_1 - r_2)^T (r_1 - r_2) = l^2
\]

Presence of constraint implies presence a force (called constraint force), that forces this constraint to hold.
Concept of Holonomic Constraint

A constraint imposed on $k$ particles (with coordinates $r_1, r_2, \ldots, r_k \in \mathbb{R}^3$) is called holonomic, if it is of the form

$$g_i(r_1, r_2, \ldots, r_k) = 0, \quad i = 1, 2, \ldots, l$$

Differentiating the constraint function $g_i(\cdot)$ with respect to time, we obtain new constraint

$$\frac{d}{dt}g_i(r_1, r_2, \ldots, r_k) = \frac{\partial g_i}{\partial r_1} \frac{d}{dt}r_1 + \cdots + \frac{\partial g_i}{\partial r_k} \frac{d}{dt}r_k = 0$$

or

$$\frac{\partial g_i}{\partial r_1} dr_1 + \cdots + \frac{\partial g_i}{\partial r_k} dr_k = 0$$
A constraint imposed on \( k \) particles (with coordinates \( r_1, r_2, \ldots, r_k \in \mathbb{R}^3 \)) is called holonomic, if it is of the form

\[
g_i(r_1, r_2, \ldots, r_k) = 0, \quad i = 1, 2, \ldots, l
\]

Differentiating the constraint function \( g_i(\cdot) \) with respect to time, we obtain new constraint

\[
\frac{d}{dt} g_i(r_1, r_2, \ldots, r_k) = \frac{\partial g_i}{\partial r_1} \frac{d}{dt} r_1 + \cdots + \frac{\partial g_i}{\partial r_k} \frac{d}{dt} r_k = 0
\]

or

\[
\frac{\partial g_i}{\partial r_1} dr_1 + \cdots + \frac{\partial g_i}{\partial r_k} dr_k = 0
\]

The constraint of the form

\[
\omega_1(r_1, \ldots r_k) dr_1 + \cdots + \omega_k(r_1, \ldots r_k) dr_k = 0
\]

is called non-holonomic if it cannot be integrated back.
Concept of Generalized Coordinates

If the system is subject to holonomic constraint then

- If system consists of \( k \) particles, then it may be possible to express their coordinates as functions of fewer than \( 3k \) variables

\[
r_1 = r_1(q_1, \ldots, q_n), \quad r_2 = r_2(q_1, \ldots, q_n), \ldots,
\]

\[
r_k = r_k(q_1, \ldots, q_n)
\]
Concept of Generalized Coordinates

If the system is subject to holonomic constraint then

• If system consists of $k$ particles, then it may be possible to express their coordinates as functions of fewer than $3k$ variables

$$r_1 = r_1(q_1, \ldots, q_n), \quad r_2 = r_2(q_1, \ldots, q_n), \quad \ldots,$$

$$r_k = r_k(q_1, \ldots, q_n)$$

• The smallest set of variables is called generalized coordinates

• This smallest number is called a number of degrees of freedom
Concept of Generalized Coordinates

If the system is subject to holonomic constraint then

- If system consists of \( k \) particles, then it may be possible to express their coordinates as functions of fewer than \( 3k \) variables

\[
r_1 = r_1(q_1, \ldots, q_n), \quad r_2 = r_2(q_1, \ldots, q_n), \quad \ldots,
\]

\[
r_k = r_k(q_1, \ldots, q_n)
\]

- The smallest set of variables is called \textit{generalized coordinates}

- This smallest number is called a \textit{number of degrees of freedom}

- If the system consists of an \textit{infinite} number of particles, then it might have \textit{finite} number of degrees of freedom
Given a system of $k$-particles and a holonomic constraint

$$g_i(r_1, r_2, \ldots, r_k) = 0, \quad i = 1, 2, \ldots, l$$

or the same

$$\frac{\partial g_i}{\partial r_1} \, dr_1 + \cdots + \frac{\partial g_i}{\partial r_k} \, dr_k = 0, \quad i = 1, 2, \ldots, l$$
Concept of Virtual Displacement

Given a system of \( k \)-particles and a holonomic constraint

\[
g_i(r_1, r_2, \ldots, r_k) = 0, \quad i = 1, 2, \ldots, l
\]

or the same

\[
\frac{\partial g_i}{\partial r_1} dr_1 + \cdots + \frac{\partial g_i}{\partial r_k} dr_k = 0, \quad i = 1, 2, \ldots, l
\]

By definition a set of infinitesimal displacements

\[
\delta r_1, \delta r_2, \ldots, \delta r_k
\]

that are consistent with the constraint, i.e.

\[
\frac{\partial g_i}{\partial r_1} \delta r_1 + \cdots + \frac{\partial g_i}{\partial r_k} \delta r_k = 0, \quad i = 1, 2, \ldots, l
\]

are called virtual displacements
Concept of Virtual Displacement

Virtual displacements of a rigid bar. Such infinitesimal motions do not destroy the constraint

\[(r_1 - r_2)^T (r_1 - r_2) = l^2\]

if \(r_1\) and \(r_2\) are perturbed

\[r_1 \rightarrow (r_1 + \delta r_1) \quad r_2 \rightarrow (r_2 + \delta r_2)\]

that is

\[((r_1 + \delta r_1) - (r_2 + \delta r_2))^T ((r_1 + \delta r_1) - (r_2 + \delta r_2)) = l^2\]
Virtual displacements of a rigid bar. Such infinitesimal motions do not destroy the constraint

\[(r_1 - r_2)^T (r_1 - r_2) = l^2\]

if \(r_1\) and \(r_2\) are perturbed

\[r_1 \rightarrow (r_1 + \delta r_1) \quad r_2 \rightarrow (r_2 + \delta r_2)\]

that is

\[(r_1 - r_2)^T (\delta r_1 - \delta r_2) = l^2\]
Principle of Virtual Work

Consider a system of $k$-particles, suppose that

- The system has a holonomic constraint, that is some of particles exposed to constraint forces $f^C_i$;
Principle of Virtual Work

Consider a system of $k$-particles, suppose that

- The system has a holonomic constraint, that is some of particles exposed to constraint forces $f^c_i$;
- There are externally applied forces $f^e_i$ to the system;
Principle of Virtual Work

Consider a system of $k$-particles, suppose that

- The system has a holonomic constraint, that is some of particles exposed to constraint forces $f_i^c$;
- There are externally applied forces $f_i^e$ to the particles;
- The system is in an equilibrium (i.e. at rest)
Principle of Virtual Work

Consider a system of $k$-particles, suppose that

- The system has a holonomic constraint, that is some of particles exposed to constraint forces $f^c_i$;
- There are externally applied forces $f^e_i$ to the particles;
- The system is in an equilibrium (i.e. at rest)

Then the total sum of all forces applied to $i^{th}$-particle is zero

$$\sum_i (f^c_i + f^e_i) = 0$$
Principle of Virtual Work

Consider a system of \( k \)-particles, suppose that

- The system has a holonomic constraint, that is some of particles exposed to constraint forces \( f^c_i \); 
- There are externally applied forces \( f^e_i \) to the particles; 
- The system is in an equilibrium (i.e. at rest)

Then the total sum of all forces applied to \( i^{th} \)-particle is zero

\[
\sum_i (f^c_i + f^e_i) = 0
\]

Then the work done by all forces applied to \( i^{th} \)-particle along each set of virtual displacement is zero, i.e.

\[
0 = \sum_i (f^c_i + f^e_i) \delta r_i = \sum_i f^c_i \delta r_i + \sum_i f^e_i \delta r_i = 0
\]