• Concept of Configuration Space
Lecture 8: Kinematics: Path and Trajectory Planning

- Concept of Configuration Space
- Path Planning
  - Potential Field Approach
  - Probabilistic Road Map Method
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Concept of Configuration Space

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- A complete specification of location of the robot is called its configuration
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- For example, for 1-link revolute arm $\mathcal{Q}$ is the set of all possible orientations of the link, i.e.

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\mathcal{Q} = S^1 \quad \text{or} \quad \mathcal{Q} = SO(2)
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- For example, for 2-link planar arm with revolute joints
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  \mathcal{Q} = S^1 \times S^1 = T^2 \quad \leftarrow \quad \text{torus}
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- For example, for 2-link planar arm with revolute joints

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Q = S^1 \times S^1 = T^2 \quad \leftarrow \quad \text{torus}
\]

- For example, for a rigid object moving on a plane

\[
Q = \{ x, y, \theta \} = \mathbb{R}^2 \times S^1
\]
Concept of Configuration Space

Given a robot with $n$-links and its configuration space,

- Denote $\mathcal{W}$ the subset of $\mathbb{R}^3$ where the robot moves. It is called \textit{workspace} of the robot.
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- The workspace \(W\) might contain obstacles \(O_i\).
- Denote \(A\) a subset of workspace \(W\), which is occupied by the robot, \(A = A(q)\).
- Introduce a subset of configuration space that is occupied by obstacles

\[
QO := \{ q \in Q : A(q) \cap O_i \neq \emptyset, \forall i \}
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Concept of Configuration Space

Given a robot with \( n \)-links and its configuration space,

- Denote \( \mathcal{W} \) the subset of \( \mathbb{R}^3 \) where the robot moves. It is called *workspace* of the robot.

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- Denote \( \mathcal{A} \) a subset of workspace \( \mathcal{W} \), which is occupied by the robot, \( \mathcal{A} = \mathcal{A}(q) \).

- Introduce a subset of configuration space that is occupied by obstacles

\[
\mathcal{QO} := \{ q \in \mathcal{Q} : \mathcal{A}(q) \cap O_i \neq \emptyset, \ \forall i \}
\]

- Then *collision-free configurations* are defined by

\[
\mathcal{Q}_{\text{free}} := \mathcal{Q} \setminus \mathcal{QO}
\]
(a) The end-effector of the robot has a form of triangle. It moves in a plane. The plane contains a rectangular obstacle.

(b) $\mathcal{O}$ is the set with the dashed boundary
(a) Two-links planar arm robot. The workspace has a single square obstacle.

(b) The configuration space and the set $QO$ occupied by the obstacle is in gray.
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Path Planning

Problem of Path Planning is the task to find a path in the configuration space $Q$

- that connects an initial configuration $q_s$ to a final configuration $q_f$
- that does not collide any obstacle as the robot traverses the path.
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Formally, the task is to find a continuous function $\gamma(\cdot)$ such that

$$\gamma : [0, 1] \rightarrow Q_{free} \quad \text{with} \quad \gamma(0) = q_s, \text{ and } \gamma(1) = q_f$$
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Common additional requirements:
- Some intermediate points $q_i$ can be given
- Smoothness of a path
- Optimality (length, curvature, etc)
Path Planning: Potential Field Approach

Basic idea:

- Treat a robot as a particle under an influence of an artificial potential field $U(\cdot)$;
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  - global minimum at $q_f \Rightarrow$ this point is attractive
  - maximum or to be $+\infty$ in the points of $QO \Rightarrow$ these points repel the robot
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- Try to find such function $U(\cdot)$ constructed in a simple form, where we can easily add or remove an obstacle and change $q_f$. The common form for $U(\cdot)$ is

$$U(q) = U_{att}(q) + \left( U_{rep}^{(1)}(q) + U_{rep}^{(2)}(q) + \cdots + U_{rep}^{(N)}(q) \right)$$
Path Planning: Probabilistic Road Map Method

Basic idea:

• Sample randomly the configuration space $Q$;
• Those samples that belong to $Q_0$ are disregarded;
Path Planning: Probabilistic Road Map Method

Basic idea:

- Sample randomly the configuration space $Q$;
- Those samples that belong to $Q_O$ are disregarded;
- Connecting Pairs of Configuration, e.g.
  - Choose the way measure the distance $d(\cdot)$ in $Q$
  - Choose $\varepsilon > 0$ and find $k$ neighbors of distance no more than $\varepsilon$ that can be connected to the current one

This step will result in fragmentation of the workspace consisting of several disjoint components
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- Make enhancement, that is, try to connect disjoint components
- Try to compute a smooth path from a family of points
Steps in constructing probabilistic roadmap
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Trajectory Planning

Trajectory is a path $\gamma : [0, 1] \rightarrow Q_{free}$ with explicit parametrization of time

$$[T_s, T_f] \ni t \mapsto \tau \in [0, 1] : q(t) = \gamma(\tau) \in Q_{free}$$
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$$[T_s, T_f] \ni t \mapsto \tau \in [0, 1] : q(t) = \gamma(\tau) \in \mathcal{Q}_{\text{free}}$$

This means that we make specifications on

- velocity $\frac{d}{dt} q(t)$ of a motion;
- acceleration $\frac{d^2}{dt^2} q(t)$ of a motion;
- jerk $\frac{d^3}{dt^3} q(t)$ of a motion;
- ...
Trajectory Planning

Trajectory is a path $\gamma : [0, 1] \rightarrow Q_{free}$ with explicit parametrization of time

$$[T_s, T_f] \ni t \rightsquigarrow \tau \in [0, 1] : q(t) = \gamma(\tau) \in Q_{free}$$

This means that we make specifications on

- velocity $\frac{dq(t)}{dt}$ of a motion;
- acceleration $\frac{d^2q(t)}{dt^2}$ of a motion;
- jerk $\frac{d^3q(t)}{dt^3}$ of a motion;
- ... 

In fact, it is common that the path is not given completely, but as a family of snapshots

$$q_s, q_1, q_2, q_3, \ldots, q_f$$

So that we have substantial freedom in generating trajectories.
Decomposition of a path into segments with fast and slow velocity profiles
Consider the $i^{th}$ joint of a robot and suppose that the specification

at time $t = t_0$ is: \[ q_i(t_0) = q_0, \quad \frac{d}{dt}q(t_0) = v_0 \]
Consider the $i^{th}$ joint of a robot and suppose that the specification

at time $t = t_0$ is: $q_i(t_0) = q_0$, $\frac{dq}{dt}q(t_0) = v_0$

at time $t = t_1$ is: $q_i(t_1) = q_1$, $\frac{dq}{dt}q(t_1) = v_1$
Trajectories for Point to Point Motion

Consider the \( i^{th} \) joint of a robot and suppose that the specification

- at time \( t = t_0 \) is: \( q_i(t_0) = q_0, \frac{d}{dt} q(t_0) = v_0 \)
- at time \( t = t_f \) is: \( q_i(t_f) = q_f, \frac{d}{dt} q(t_f) = v_f \)

In addition, we might be given constraints of accelerations

\[
\frac{d^2}{dt^2} q(t_0) = \alpha_0 \quad \frac{d^2}{dt^2} q(t_f) = \alpha_f
\]
Trajectories for Point to Point Motion

Consider the $i^{th}$ joint of a robot and suppose that the specification at time $t = t_0$ is:
\[ q_i(t_0) = q_0, \quad \frac{d}{dt} q(t_0) = v_0 \]

at time $t = t_f$ is:
\[ q_i(t_f) = q_f, \quad \frac{d}{dt} q(t_f) = v_f \]

In addition, we might be given constraints of accelerations
\[ \frac{d^2}{dt^2} q(t_0) = \alpha_0, \quad \frac{d^2}{dt^2} q(t_f) = \alpha_f \]

If we choose to generate a polynomial
\[ q(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_m t^m \]

that will satisfy the interpolation constraints, what degree this polynomial should be chosen?
Trajectories for Point to Point Motion

The interpolation constraints

at time $t = t_0$ is: $q_i(t_0) = q_0$, $\frac{d}{dt}q(t_0) = v_0$

at time $t = t_f$ is: $q_i(t_f) = q_f$, $\frac{d}{dt}q(t_f) = v_f$

for the $3^{rd}$-order polynomial

$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$, $\frac{d}{dt}q(t) = a_1 + 2a_2 t + 3a_3 t^2$
Trajectories for Point to Point Motion

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$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3, \quad \frac{d}{dt} q(t) = a_1 + 2a_2 t + 3a_3 t^2$

are

$q_0 = a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3$

$v_0 = a_1 + 2a_2 t_0 + 3a_3 t_0^2$

$q_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3$

$v_f = a_1 + 2a_2 t_f + 3a_3 t_f^2$
Trajectories for Point to Point Motion

The equations

\[
\begin{align*}
q_0 &= a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3 \\
v_0 &= a_1 + 2a_2 t_0 + 3a_3 t_0^2 \\
q_f &= a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 \\
v_f &= a_1 + 2a_2 t_f + 3a_3 t_f^2
\end{align*}
\]

written in matrix form are

\[
\begin{bmatrix}
q_0 \\
v_0 \\
q_f \\
v_f
\end{bmatrix}
= \begin{bmatrix}
1 & t_0 & t_0^2 & t_0^3 \\
0 & 1 & 2t_0 & 3t_0^2 \\
1 & t_f & t_f^2 & t_f^3 \\
0 & 1 & 2t_f & 3t_f^2
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{bmatrix}
\]
Trajectories for Point to Point Motion

The equations

\[ q_0 = a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3 \]
\[ v_0 = a_1 + 2a_2 t_0 + 3a_3 t_0^2 \]
\[ q_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 \]
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0 & 1 & 2t_f & 3t_f^2
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{bmatrix}
\]

What is the determinant of this matrix?
The parameters for interpolation

\[ t = 0 \text{ and } t_f = 1, \quad q_0 = 10 \text{ and } q_f = -20, \quad v_0 = v_f = 0 \]

what is wrong with the trajectory?
Consider the $i^{th}$ joint of a robot and suppose that the specification

at time $t = t_0$ is: $q_i(t_0) = q_0, \frac{dq}{dt}(t_0) = v_0$

at time $t = t_f$ is: $q_i(t_f) = q_f, \frac{dq}{dt}(t_f) = v_f$

and additional constraints of accelerations

$\frac{d^2}{dt^2} q(t_0) = \alpha_0, \quad \frac{d^2}{dt^2} q(t_f) = \alpha_f$
Trajectories for Point to Point Motion

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and additional constraints of accelerations

$\frac{d^2}{dt^2} q(t_0) = \alpha_0$ \quad $\frac{d^2}{dt^2} q(t_f) = \alpha_f$

To find interpolating polynomial we need to choose a polynomial of order $\geq 5$

$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$
The parameters for interpolation

\[ t = 0 \text{ and } t_f = 2, \quad q_0 = 0 \text{ and } q_f = 20, \quad v_0 = v_f = 0 \]
Interpolation by LSPB: Linear segments with parabolic blends