Lecture 7: Kinematics: Velocity Kinematics - the Jacobian

- Manipulator Jacobian

Lecture 7: Kinematics: Velocity Kinematics - the Jacobian

- Manipulator Jacobian
- Analytical Jacobian

Lecture 7: Kinematics: Velocity Kinematics - the Jacobian

- Manipulator Jacobian
- Analytical Jacobian
- Singularities

Lecture 7: Kinematics: Velocity Kinematics - the Jacobian

- Manipulator Jacobian
- Analytical Jacobian
- Singularities
- Inverse Velocity and Manipulability

Concept of the Manipulator Jacobian

Given an n-link manipulator with joint variables $q_1, \ldots, q_n$

- Let $T_n^0(q)$ is the homogeneous transformation between the end-effector and base frames

$$T_n^0(q) = \begin{bmatrix} R_n^0(q) & o_n^0(q) \\ 0 & 1 \end{bmatrix}, \quad q = \left[q_1, \ldots, q_n\right]^T$$

so that $\forall\, p$ with coordinates p^n its coordinates in the base frame are

$$p^0 = R_n^0\, p^n + o_n^0$$

Concept of the Manipulator Jacobian

Given an n-link manipulator with joint variables $q_1, \ldots, q_n$

- Let $T_n^0(q)$ is the homogeneous transformation between the end-effector and base frames

$$T_n^0(q) = \begin{bmatrix} R_n^0(q) & o_n^0(q) \\ 0 & 1 \end{bmatrix}, \quad q = \left[q_1, \ldots, q_n\right]^T$$

so that $\forall\, p$ with coordinates p^n its coordinates in the base frame are

$$p^0 = R_n^0\, p^n + o_n^0$$

- As the robot moves, joint variables become functions of time

$$t \;\longrightarrow\; q(t) = \left[q_1(t), \ldots, q_n(t)\right]^T$$

so that

$$p^0(t) = R_n^0(q(t))\, p^n + o_n^0(q(t))$$

Concept of the Manipulator Jacobian

- We have already seen how to compute $\frac{d}{dt}p^0(t)$ from

$$p^0(t) = R_n^0(q(t))\, p^n + o_n^0(q(t))$$

Concept of the Manipulator Jacobian

- We have already seen how to compute $\frac{d}{dt}p^0(t)$ from

$$p^0(t) = R_n^0(q(t))\, p^n + o_n^0(q(t))$$

- It is

$$\frac{d}{dt}p^0(t) = \frac{d}{dt}\left[R_n^0(q(t))\right] p^n + \frac{d}{dt}\left[o_n^0(q(t))\right]$$

$$= S(\omega_{0,n}^0(t))R_n^0(q(t))p^n + v_n^0(t) = \omega_{0,n}^0(t) \times r(t) + v_n^0(t)$$

with $r(t) = p - o_n(t)$

Concept of the Manipulator Jacobian

- We have already seen how to compute $\frac{d}{dt}p^0(t)$ from

$$p^0(t) = R_n^0(q(t))\, p^n + o_n^0(q(t))$$

- It is

$$\frac{d}{dt}p^0(t) = \frac{d}{dt}\left[R_n^0(q(t))\right] p^n + \frac{d}{dt}\left[o_n^0(q(t))\right]$$

$$= S(\omega_{0,n}^0(t))R_n^0(q(t))p^n + v_n^0(t) = \omega_{0,n}^0(t) \times r(t) + v_n^0(t)$$

with $r(t) = p - o_n(t)$

- Therefore, to compute velocity of any point in the end-effector frame, it suffices to know
  - an angular velocity $\omega_{0,n}^0(t)$ of the end-effector frame
  - a linear velocity $v_n^0(t)$ of the the end-effector frame origin

Concept of the Manipulator Jacobian

Given

- an n-link manipulator with joint variables $q_1, \ldots, q_n$
- its particular motion $q(t) = \left[q_1(t), \ldots, q_n(t)\right]^T$

What do the functions $\omega_{0,n}^0(t)$ and $v_n^0(t)$ depend on?

Concept of the Manipulator Jacobian

Given

- an n-link manipulator with joint variables $q_1, \ldots, q_n$
- its particular motion $q(t) = \left[q_1(t), \ldots, q_n(t)\right]^T$

What do the functions $\omega_{0,n}^0(t)$ and $v_n^0(t)$ depend on?

It is suggested to search them as

$$v_n^0(t) = J_v(q(t))\,\frac{d}{dt}q(t) \qquad \omega_{0,n}^0(t) = J_\omega(q(t))\,\frac{d}{dt}q(t)$$

Concept of the Manipulator Jacobian

Given

- an n-link manipulator with joint variables $q_1, \ldots, q_n$
- its particular motion $q(t) = \left[q_1(t), \ldots, q_n(t)\right]^T$

What do the functions $\omega_{0,n}^0(t)$ and $v_n^0(t)$ depend on?

It is suggested to search them as

$$v_n^0(t) = J_v(q(t))\,\tfrac{d}{dt}q(t) \qquad \omega_{0,n}^0(t) = J_\omega(q(t))\,\tfrac{d}{dt}q(t)$$

The $6 \times n$-matrix function $J(\cdot)$ defined by

$$\xi(t) = \begin{bmatrix} v_n^0(t) \\ \omega_{0,n}^0(t) \end{bmatrix} = J(q(t))\,\tfrac{d}{dt}q(t) = \begin{bmatrix} J_v(q(t)) \\ J_\omega(q(t)) \end{bmatrix} \tfrac{d}{dt}q(t)$$

is the manipulator Jacobian; $\xi(t)$ is a vector of body velocities

Angular Velocity

Given n-moving frames with the same origins as for fixed one

$$R_n^0(t) = R_1^0(t) R_2^1(t) \cdots R_n^{n-1}(t) \Rightarrow \frac{d}{dt} R_n^0(t) = S(\omega_{0,n}^0(t)) R_n^0(t)$$

$$\begin{aligned} \omega_{0,n}^0(t) &= \omega_{0,1}^0(t) + \omega_{1,2}^0(t) + \omega_{2,3}^0(t) + \cdots + \omega_{n-1,n}^0(t) \\ &= \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1 + R_2^0 \omega_{2,3}^2 + \cdots + R_{n-1}^0 \omega_{n-1,n}^{n-1} \end{aligned}$$

Angular Velocity

Given n-moving frames with the same origins as for fixed one

$$R_n^0(t) = R_1^0(t) R_2^1(t) \cdots R_n^{n-1}(t) \;\Rightarrow\; \frac{d}{dt} R_n^0(t) = S(\omega_{0,n}^0(t)) R_n^0(t)$$

$$\begin{aligned} \omega_{0,n}^0(t) &= \omega_{0,1}^0(t) + \omega_{1,2}^0(t) + \omega_{2,3}^0(t) + \cdots + \omega_{n-1,n}^0(t) \\ &= \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1 + R_2^0 \omega_{2,3}^2 + \cdots + R_{n-1}^0 \omega_{n-1,n}^{n-1} \end{aligned}$$

If i^{th}-joint is revolute ($\rho_i = 1$), then

- axis of rotation coincides with z_i;
- angular velocity is $\omega_{i-1,i}^{i-1} = \dot{q}_i(t) \cdot \vec{k}$

Angular Velocity

Given n-moving frames with the same origins as for fixed one

$$R_n^0(t) = R_1^0(t) R_2^1(t) \cdots R_n^{n-1}(t) \;\Rightarrow\; \frac{d}{dt} R_n^0(t) = S(\omega_{0,n}^0(t)) R_n^0(t)$$

$$\begin{aligned} \omega_{0,n}^0(t) &= \omega_{0,1}^0(t) + \omega_{1,2}^0(t) + \omega_{2,3}^0(t) + \cdots + \omega_{n-1,n}^0(t) \\ &= \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1 + R_2^0 \omega_{2,3}^2 + \cdots + R_{n-1}^0 \omega_{n-1,n}^{n-1} \end{aligned}$$

If i^{th}-joint is revolute ($\rho_i = 1$), then

- axis of rotation coincides with z_i;
- angular velocity is $\omega_{i-1,i}^{i-1} = \dot{q}_i(t) \cdot \vec{k}$

If i^{th}-joint is prismatic ($\rho_i = 0$), then angular velocity $\omega_{i-1,i}^{i-1} = 0$

Angular Velocity

Given n-moving frames with the same origins as for fixed one

$$R_n^0(t) = R_1^0(t) R_2^1(t) \cdots R_n^{n-1}(t) \Rightarrow \frac{d}{dt} R_n^0(t) = S(\omega_{0,n}^0(t)) R_n^0(t)$$

$$\begin{aligned} \omega_{0,n}^0(t) &= \omega_{0,1}^0(t) + \omega_{1,2}^0(t) + \omega_{2,3}^0(t) + \cdots + \omega_{n-1,n}^0(t) \\ &= \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1 + R_2^0 \omega_{2,3}^2 + \cdots + R_{n-1}^0 \omega_{n-1,n}^{n-1} \end{aligned}$$

If i^{th}-joint is revolute ($\rho_i = 1$), then

- axis of rotation coincides with z_i;
- angular velocity is $\omega_{i-1,i}^{i-1} = \dot{q}_i(t) \cdot \vec{k}$

If i^{th}-joint is prismatic ($\rho_i = 0$), then angular velocity $\omega_{i-1,i}^{i-1} = 0$

Therefore

$$\begin{aligned} \omega_{0,n}^0(t) &= \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1 + R_2^0 \omega_{2,3}^2 + \cdots + R_{n-1}^0 \omega_{n-1,n}^{n-1} \\ &= \rho_1 \dot{q}_1 \vec{k} + \rho_2 R_1^0 \dot{q}_2 \vec{k} + \cdots + \rho_n R_{n-1}^0 \dot{q}_n \vec{k} \end{aligned}$$

Angular Velocity

Given n-moving frames with the same origins as for fixed one

$$R_n^0(t) = R_1^0(t) R_2^1(t) \cdots R_n^{n-1}(t) \;\Rightarrow\; \frac{d}{dt} R_n^0(t) = S(\omega_{0,n}^0(t)) R_n^0(t)$$

$$\begin{aligned} \omega_{0,n}^0(t) &= \omega_{0,1}^0(t) + \omega_{1,2}^0(t) + \omega_{2,3}^0(t) + \cdots + \omega_{n-1,n}^0(t) \\ &= \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1 + R_2^0 \omega_{2,3}^2 + \cdots + R_{n-1}^0 \omega_{n-1,n}^{n-1} \end{aligned}$$

If i^{th}-joint is revolute ($\rho_i = 1$), then

- axis of rotation coincides with z_i;
- angular velocity is $\omega_{i-1,i}^{i-1} = \dot{q}_i(t) \cdot \vec{k}$

If i^{th}-joint is prismatic ($\rho_i = 0$), then angular velocity $\omega_{i-1,i}^{i-1} = 0$

Therefore

$$\begin{aligned} \omega_{0,n}^0(t) &= \rho_1 \dot{q}_1 \vec{k} + \rho_2 R_1^0 \dot{q}_2 \vec{k} + \cdots + \rho_n R_{n-1}^0 \dot{q}_n \vec{k} \\ &= \sum_{i=1}^{n} \rho_i \dot{q}_i z_{i-1}^0, \qquad z_{i-1}^0 = R_{i-1}^0 \vec{k} \end{aligned}$$

Angular Velocity

Given n-moving frames with the same origins as for fixed one

$$R_n^0(t) = R_1^0(t) R_2^1(t) \cdots R_n^{n-1}(t) \;\Rightarrow\; \frac{d}{dt} R_n^0(t) = S(\omega_{0,n}^0(t)) R_n^0(t)$$

$$\begin{aligned} \omega_{0,n}^0(t) &= \omega_{0,1}^0(t) + \omega_{1,2}^0(t) + \omega_{2,3}^0(t) + \cdots + \omega_{n-1,n}^0(t) \\ &= \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1 + R_2^0 \omega_{2,3}^2 + \cdots + R_{n-1}^0 \omega_{n-1,n}^{n-1} \end{aligned}$$

If i^{th}-joint is revolute ($\rho_i = 1$), then

- axis of rotation coincides with z_i;
- angular velocity is $\omega_{i-1,i}^{i-1} = \dot{q}_i(t) \cdot \vec{k}$

If i^{th}-joint is prismatic ($\rho_i = 0$), then angular velocity $\omega_{i-1,i}^{i-1} = 0$

Therefore

$$\omega_{0,n}^0(t) = \sum_{i=1}^{n} \rho_i \dot{q}_i z_{i-1}^0 = \left[\rho_1 z_0^0,\ \rho_2 z_1^0,\ \ldots,\ \rho_n z_{n-1}^0\right] \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

Angular Velocity

Given n-moving frames with the same origins as for fixed one

$$R_n^0(t) = R_1^0(t) R_2^1(t) \cdots R_n^{n-1}(t) \;\Rightarrow\; \frac{d}{dt} R_n^0(t) = S(\omega_{0,n}^0(t)) R_n^0(t)$$

$$\begin{aligned} \omega_{0,n}^0(t) &= \omega_{0,1}^0(t) + \omega_{1,2}^0(t) + \omega_{2,3}^0(t) + \cdots + \omega_{n-1,n}^0(t) \\ &= \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1 + R_2^0 \omega_{2,3}^2 + \cdots + R_{n-1}^0 \omega_{n-1,n}^{n-1} \end{aligned}$$

If i^{th}-joint is revolute ($\rho_i = 1$), then

- axis of rotation coincides with z_i;
- angular velocity is $\omega_{i-1,i}^{i-1} = \dot{q}_i(t) \cdot \vec{k}$

If i^{th}-joint is prismatic ($\rho_i = 0$), then angular velocity $\omega_{i-1,i}^{i-1} = 0$

Therefore

$$\omega_{0,n}^0(t) = \sum_{i=1}^{n} \rho_i \dot{q}_i z_{i-1}^0 = \underbrace{\left[\rho_1 z_0^0,\ \rho_2 z_1^0,\ \ldots,\ \rho_n z_{n-1}^0\right]}_{= J_\omega} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

Linear Velocity

The linear velocity $\boldsymbol{v_n^0(t)}$ of the end-effector is

the time-derivative of $o_n^0(t)$ and $\boldsymbol{v_n^0(t)} \equiv \mathbf{0}$ if $\dot{q} \equiv \mathbf{0}$

Linear Velocity

The linear velocity $v_n^0(t)$ of the end-effector is

the time-derivative of $o_n^0(t)$ and $v_n^0(t) \equiv 0$ if $\dot{q} \equiv 0$

Therefore there are functions $J_{v_1}(q, \dot{q})$, ..., $J_{v_n}(q, \dot{q})$ such that

$$v_n^0(t) \;=\; \frac{d}{dt} o_n^0(t) = J_{v_1}(\cdot)\dot{q}_1 + J_{v_2}(\cdot)\dot{q}_2 + \cdots + J_{v_n}(\cdot)\dot{q}_n$$

Linear Velocity

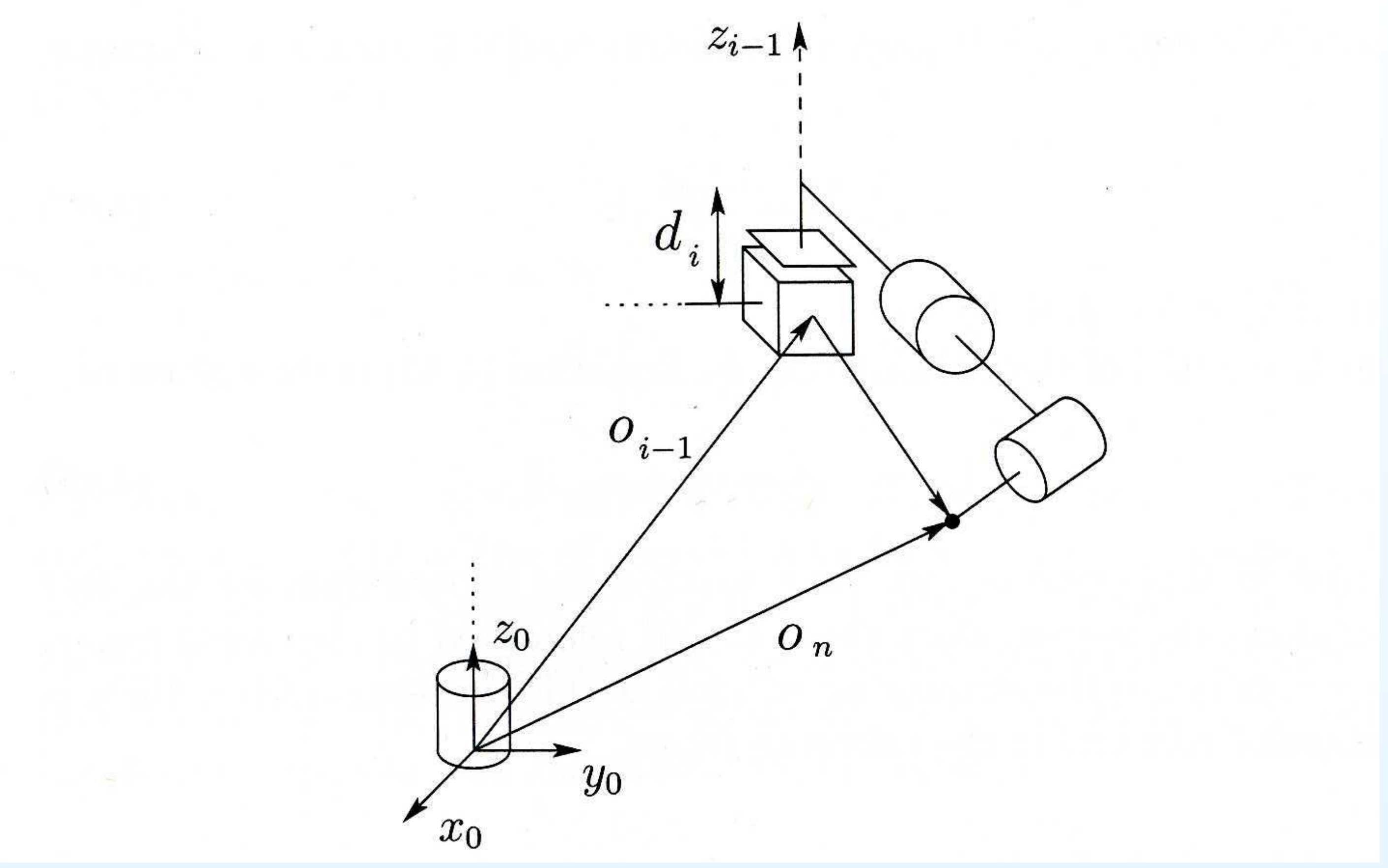
The linear velocity $v_n^0(t)$ of the end-effector is

the time-derivative of $o_n^0(t)$ and $v_n^0(t) \equiv 0$ if $\dot{q} \equiv 0$

Therefore there are functions $J_{v_1}(q, \dot{q}), \ldots, J_{v_n}(q, \dot{q})$ such that

$$\begin{aligned} v_n^0(t) &= \tfrac{d}{dt} o_n^0(t) = J_{v_1}(\cdot)\dot{q}_1 + J_{v_2}(\cdot)\dot{q}_2 + \cdots + J_{v_n}(\cdot)\dot{q}_n \\ &= \left[\frac{\partial}{\partial q_1} o_n^0(t)\right]\dot{q}_1 + \left[\frac{\partial}{\partial q_2} o_n^0(t)\right]\dot{q}_2 + \cdots + \left[\frac{\partial}{\partial q_n} o_n^0(t)\right]\dot{q}_n \end{aligned}$$

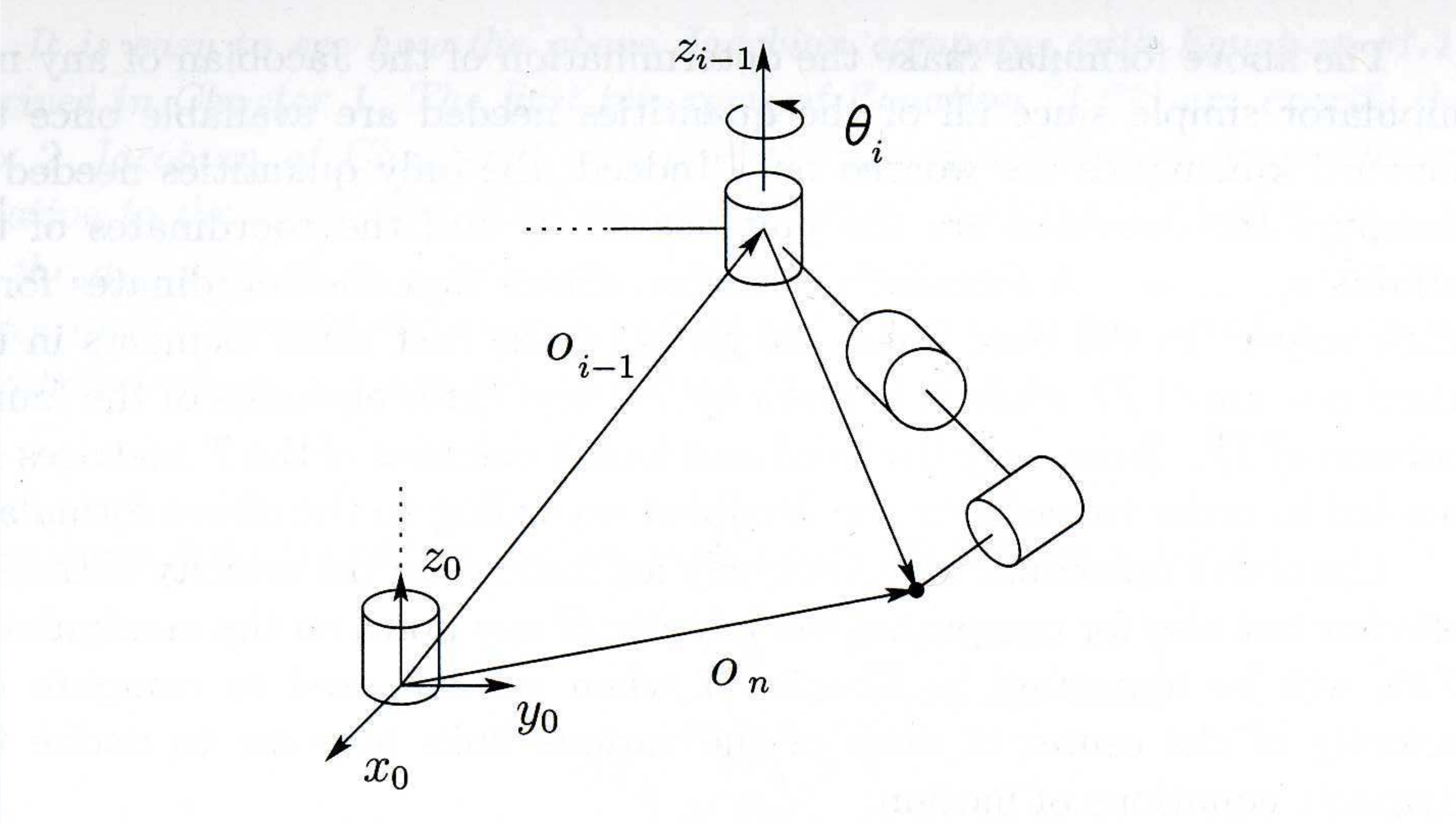
Linear Velocity



A prismatic joint i: translation along the z_{i-1}-axis with velocity $\frac{d}{dt}d_i$ when all other joints kept fixed results in

$$\frac{d}{dt}o_n^0(t) = \frac{d}{dt}d_i(t)R_{i-1}^0 k = \frac{d}{dt}d_i(t)\, z_{i-1}^0 = J_{v_i}\dot{d}_i$$

Linear Velocity



A revolute joint i: rotation along the z_{i-1}-axis with angular velocity $\frac{d}{dt}\theta_i z_{i-1}$ when all other joints kept fixed results in

$$\frac{d}{dt}o_n^0(t) = \omega \times r = \left[\dot{\theta} z_{i-1}\right] \times [o_n - o_{i-1}] = J_{v_i}\dot{\theta}_i$$

Linear Velocity

The linear velocity $v_n^0(t)$ of the end-effector is

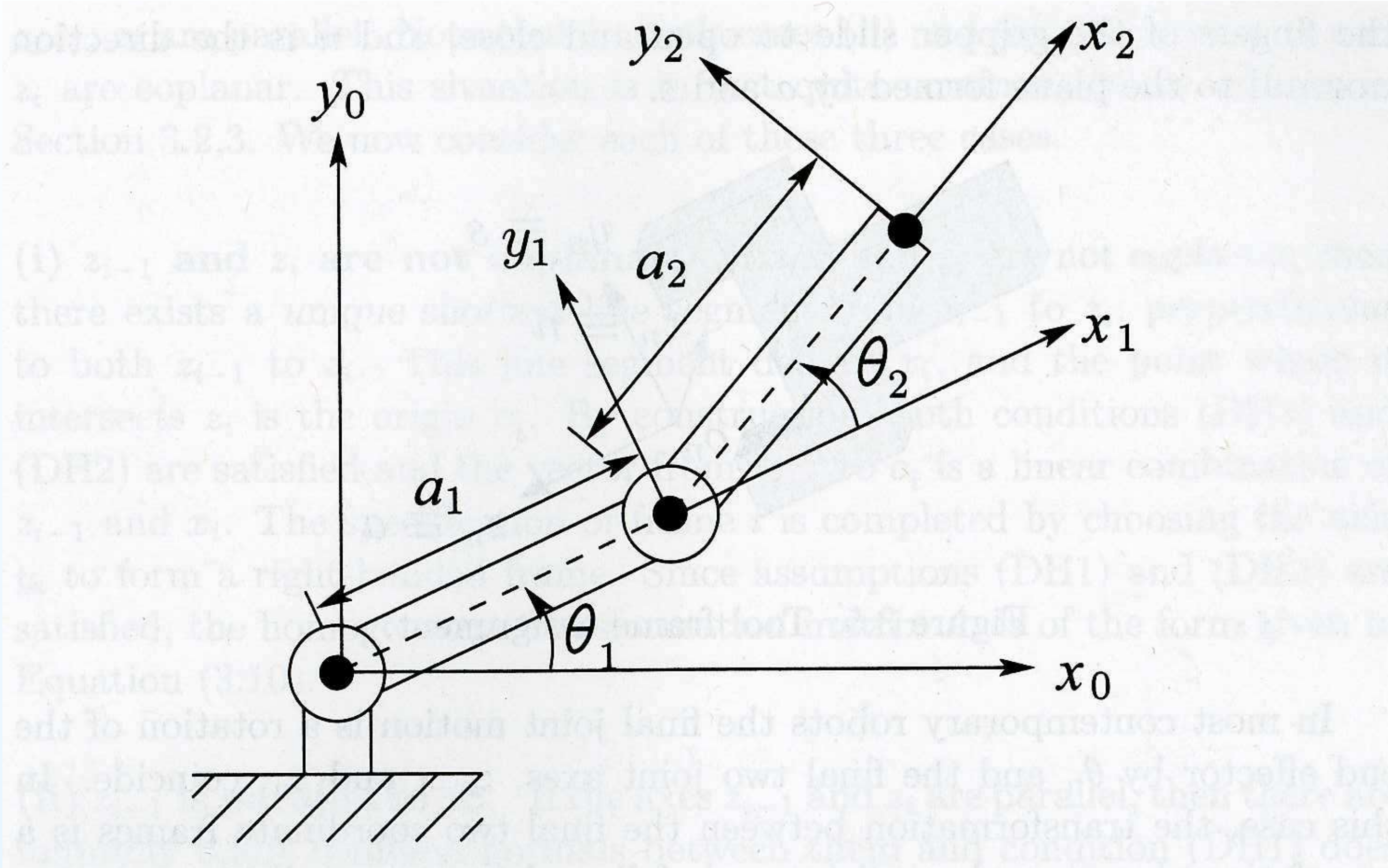
the time-derivative of $o_n^0(t)$ and $v_n^0(t) \equiv 0$ if $\dot{q} \equiv 0$

Therefore there are functions $J_{v_1}(q,\dot{q}), \ldots, J_{v_n}(q,\dot{q})$ such that

$$\begin{aligned} v_n^0(t) &= \tfrac{d}{dt} o_n^0(t) = J_{v_1}(\cdot)\dot{q}_1 + J_{v_2}(\cdot)\dot{q}_2 + \cdots + J_{v_n}(\cdot)\dot{q}_n \\ &= \left[\frac{\partial}{\partial q_1} o_n^0(t)\right]\dot{q}_1 + \left[\frac{\partial}{\partial q_2} o_n^0(t)\right]\dot{q}_2 + \cdots + \left[\frac{\partial}{\partial q_n} o_n^0(t)\right]\dot{q}_n \end{aligned}$$

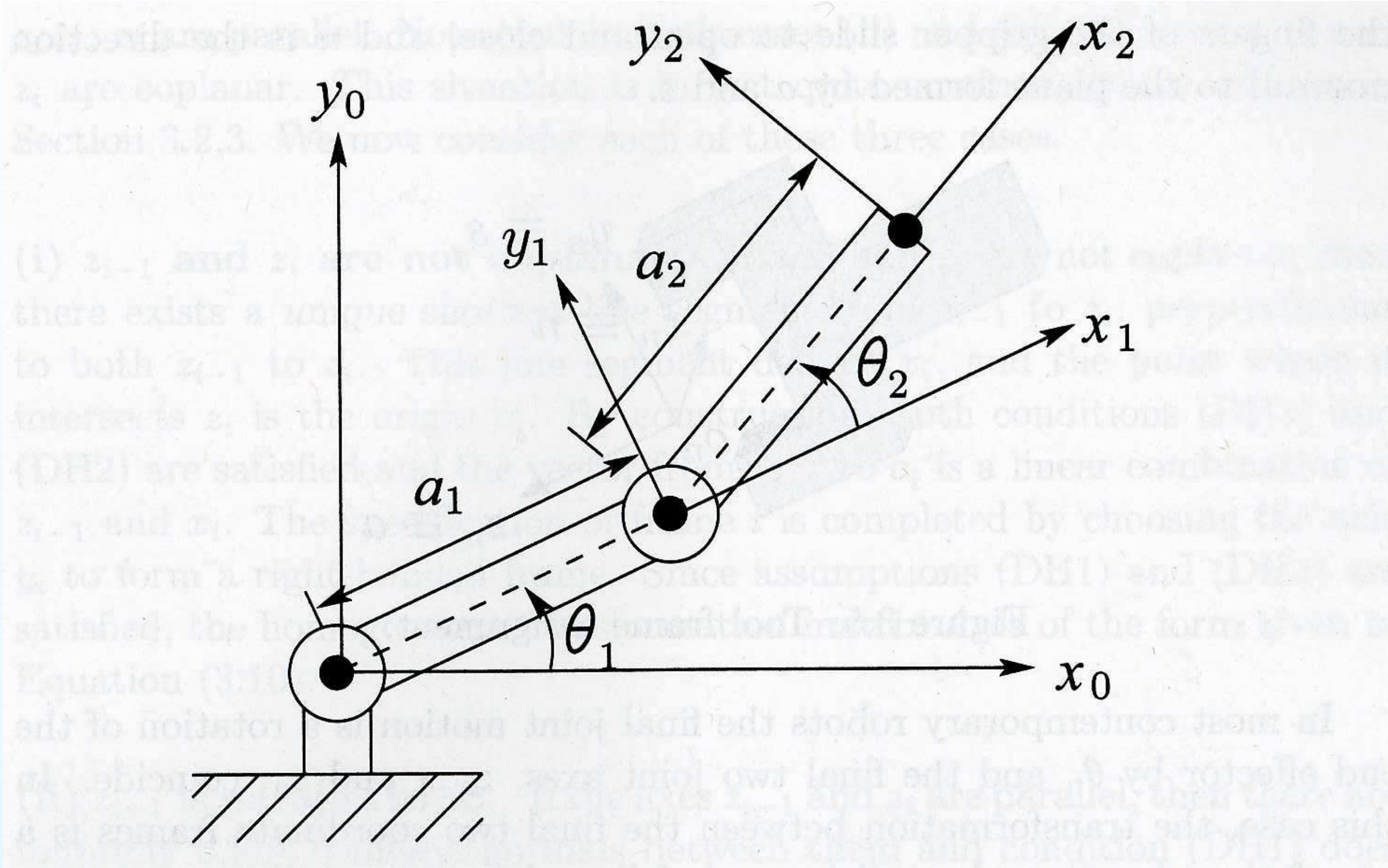
$$J_{v_i} = \begin{cases} z_{i-1}^0, & \text{for prismatic joint} \\ z_{i-1}^0 \times \left[o_n^0 - o_{i-1}^0\right], & \text{for revolute joint} \end{cases}$$

Jacobian for Example 3.1



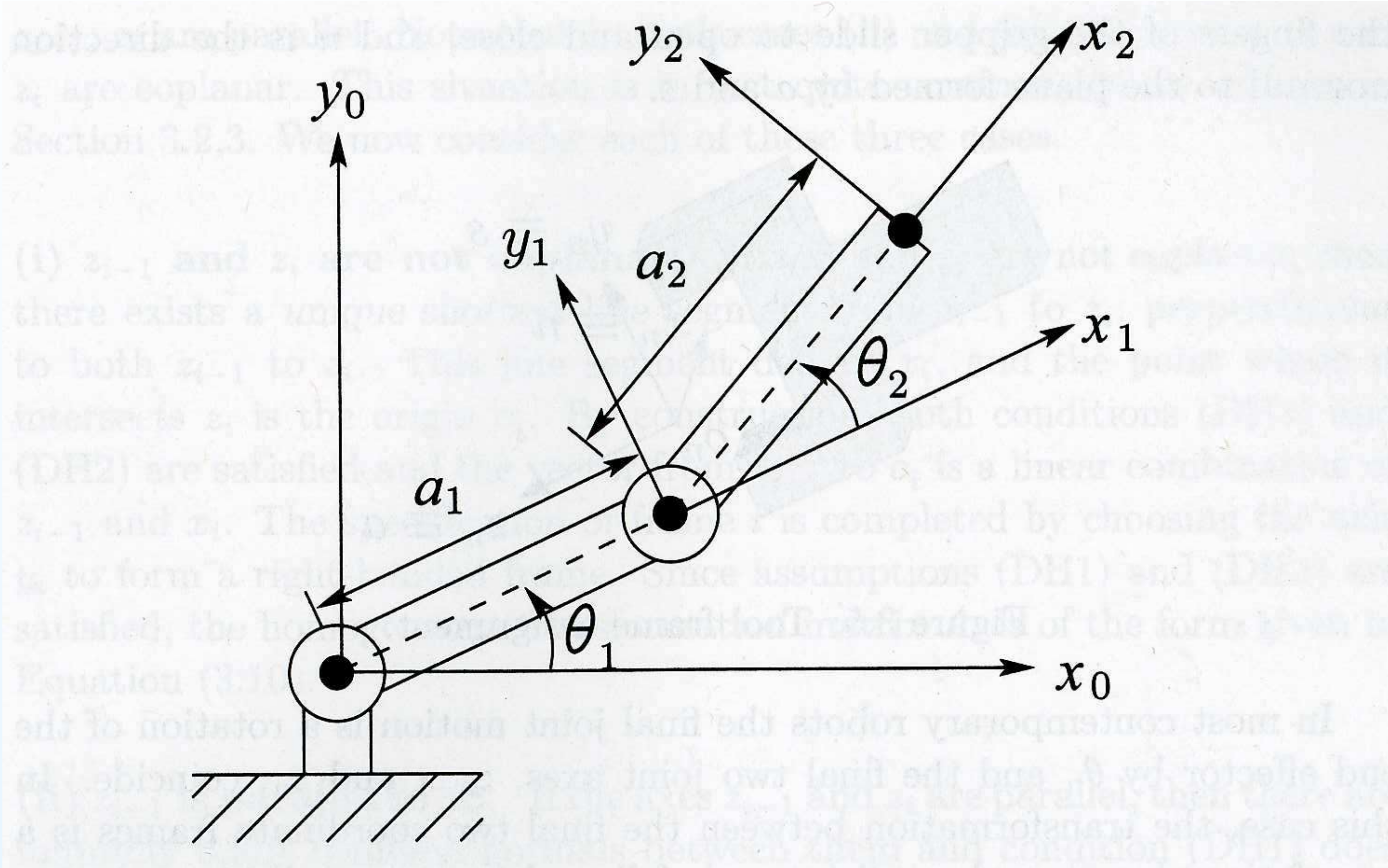
$$\begin{bmatrix} v_n^0(t) \\ \omega_{0,n}^0(t) \end{bmatrix} = \begin{bmatrix} J_v(q(t)) \\ J_\omega(q(t)) \end{bmatrix} \begin{bmatrix} \dot q_1 \\ \dot q_2 \end{bmatrix}$$

Jacobian for Example 3.1



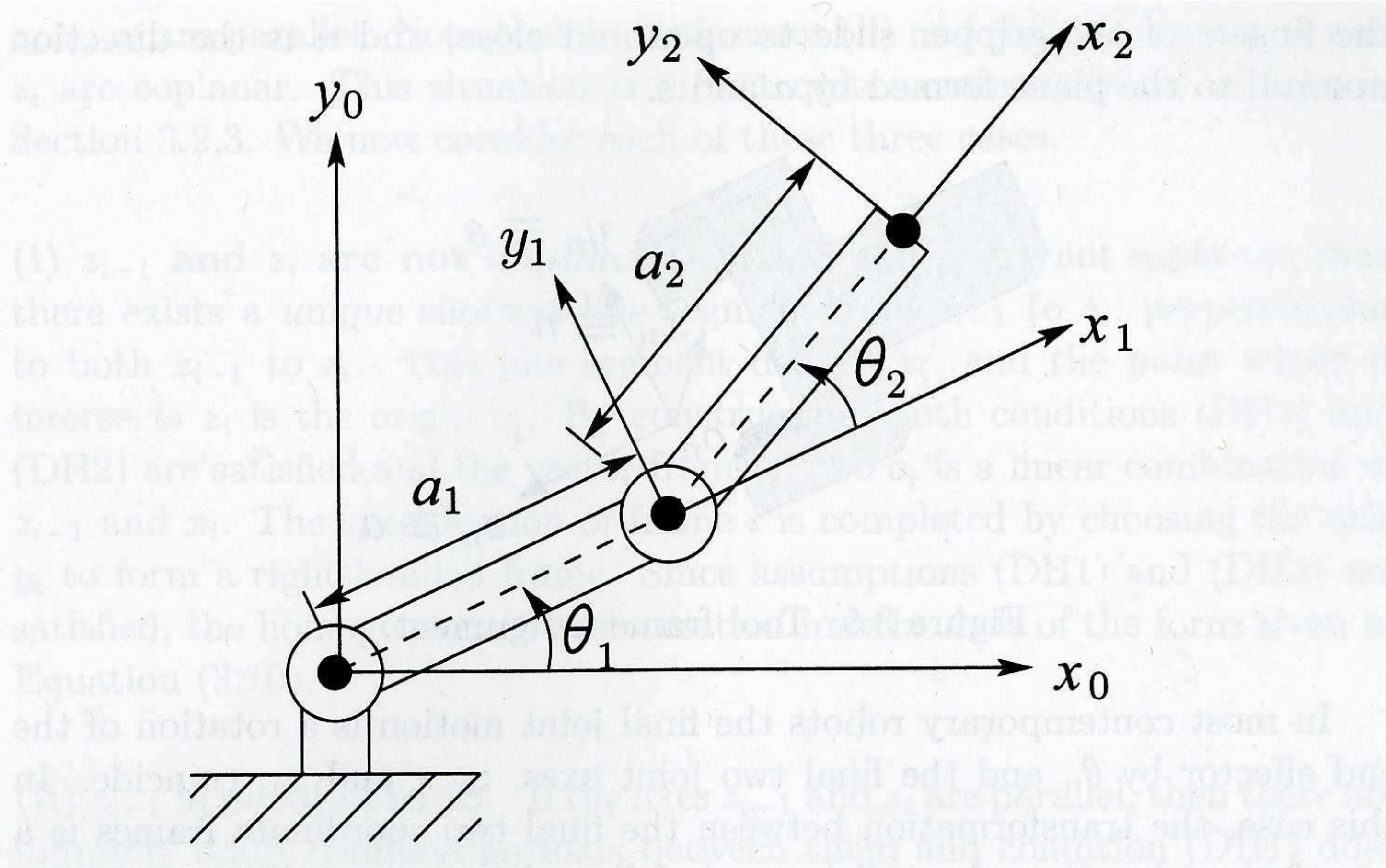
$$\begin{bmatrix} v_n^0(t) \\ \omega_{0,n}^0(t) \end{bmatrix} = \begin{bmatrix} J_{v_1}(q(t)) & J_{v_2}(q(t)) \\ J_{\omega_1}(q(t)) & J_{\omega_2}(q(t)) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

Jacobian for Example 3.1



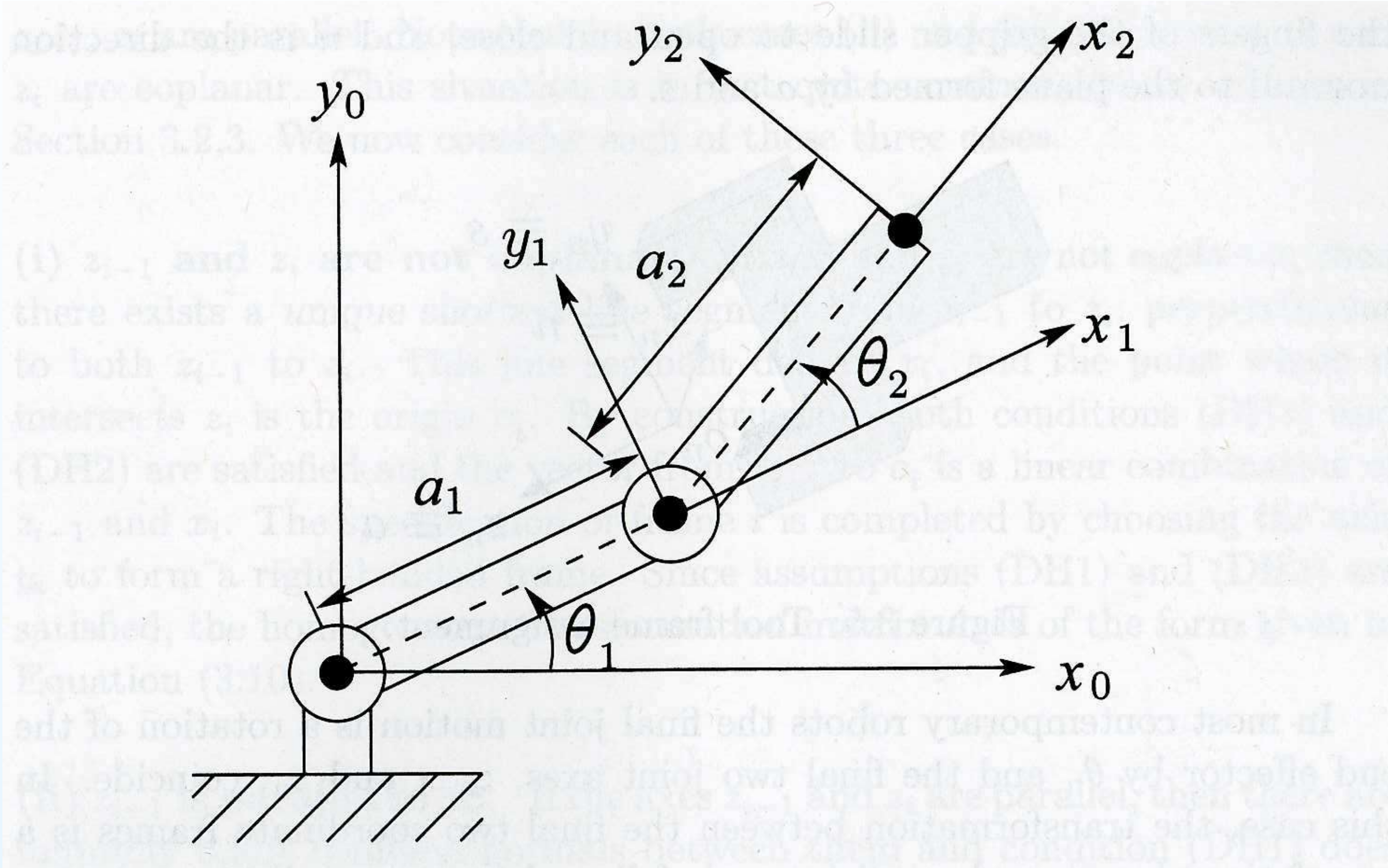
$$
\begin{bmatrix} v_n^0(t) \\ \omega_{0,n}^0(t) \end{bmatrix} = \begin{bmatrix} z_0 \times (o_2 - o_0) & J_{v_2}(q(t)) \\ J_{\omega_1}(q(t)) & J_{\omega_2}(q(t)) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}
$$

Jacobian for Example 3.1



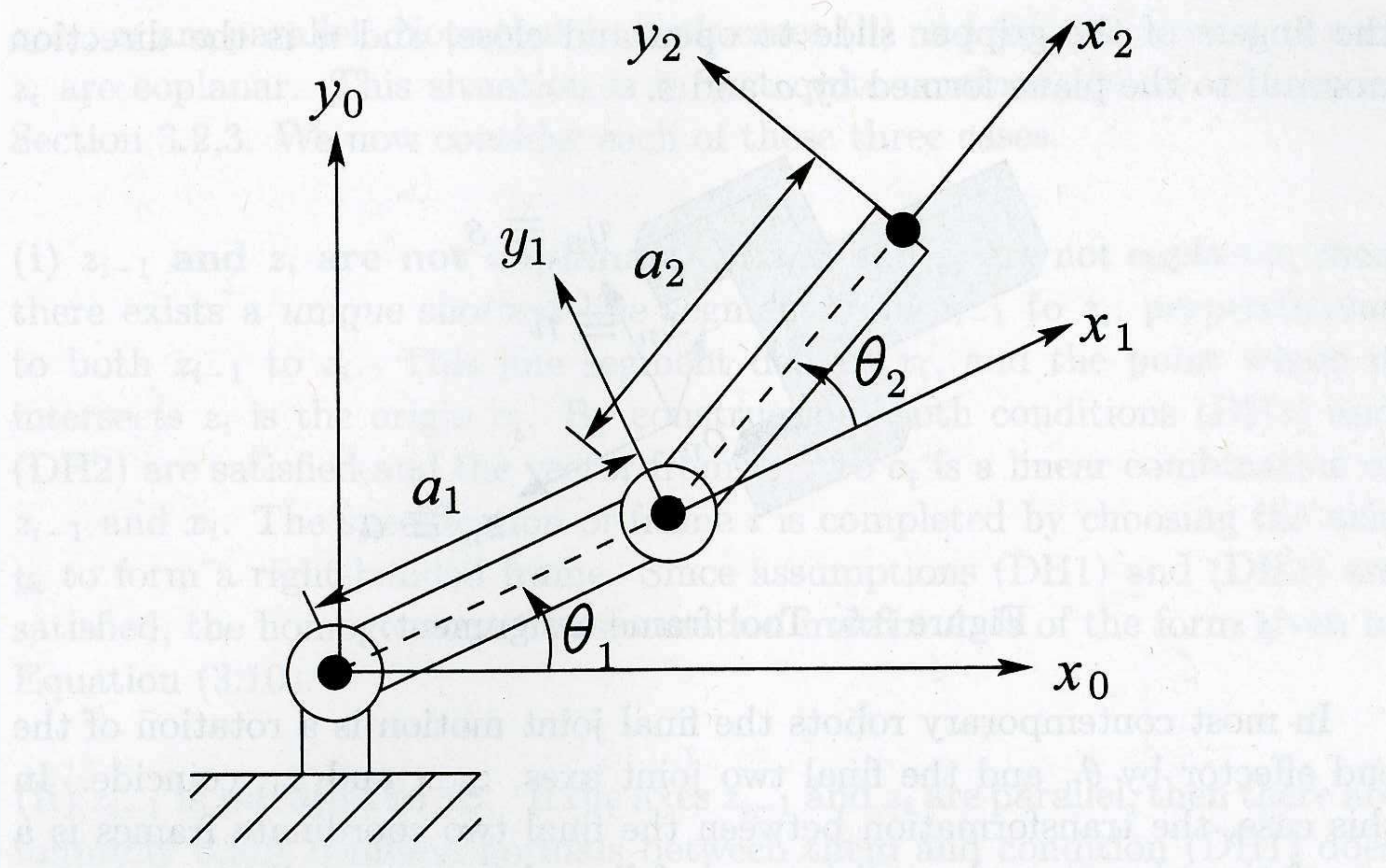
$$\begin{bmatrix} v_n^0(t) \\ \omega_{0,n}^0(t) \end{bmatrix} = \begin{bmatrix} z_0 \times (o_2 - o_0) & z_1 \times (o_2 - o_1) \\ J_{\omega_1}(q(t)) & J_{\omega_2}(q(t)) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

Jacobian for Example 3.1



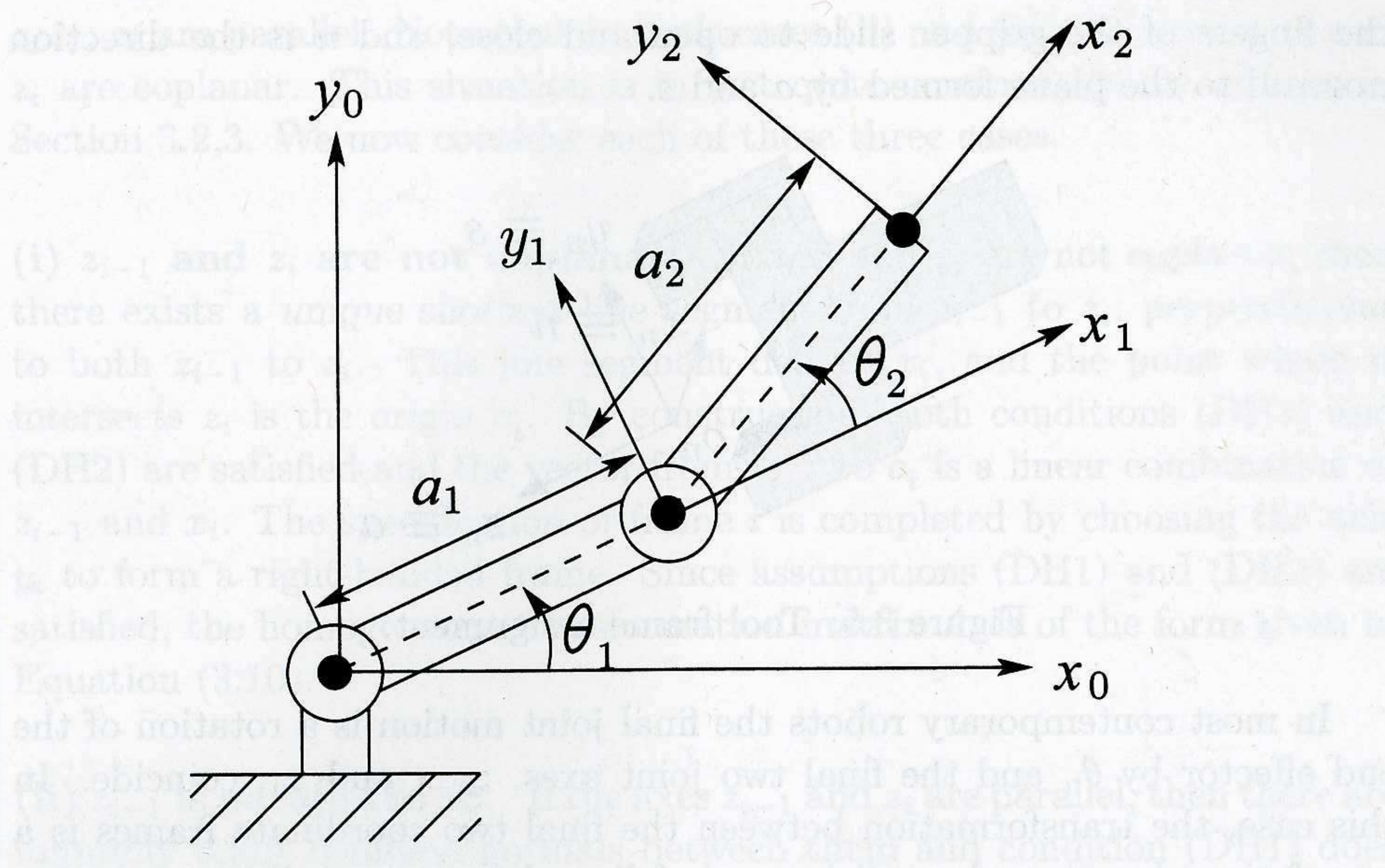
$$\begin{bmatrix} v_n^0(t) \\ \omega_{0,n}^0(t) \end{bmatrix} = \begin{bmatrix} z_0 \times (o_2 - o_0) & z_1 \times (o_2 - o_1) \\ z_0 & J_{\omega_2} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

Jacobian for Example 3.1



$$\begin{bmatrix} v_n^0(t) \\ \omega_{0,n}^0(t) \end{bmatrix} = \begin{bmatrix} z_0 \times (o_2 - o_0) & z_1 \times (o_2 - o_1) \\ z_0 & z_1 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

Jacobian for Example 3.1



$$z_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \; o_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix}, \; o_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix}$$

Lecture 7: Kinematics: Velocity Kinematics - the Jacobian

- Manipulator Jacobian
- Analytical Jacobian
- Singularities
- Inverse Velocity and Manipulability

Analytic Jacobian

Given a robot and a homogeneous transformation

$$\boldsymbol{T}_n^0(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{R}_n^0(\boldsymbol{q}) & \boldsymbol{o}_n^0(\boldsymbol{q}) \\ \boldsymbol{0} & \boldsymbol{1} \end{bmatrix}, \quad \boldsymbol{q} = [\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n]$$

there are several ways to represent rotational matrix $\boldsymbol{R}_n^0(\boldsymbol{q})$, e.g.

$$\boldsymbol{R} = \boldsymbol{R}(\phi, \theta, \psi) = \boldsymbol{R}_{z,\phi} \boldsymbol{R}_{y,\theta} \boldsymbol{R}_{z,\psi}$$

with (ϕ, θ, ψ) being the Euler angles (ZYZ-parametrization).

Analytic Jacobian

Given a robot and a homogeneous transformation

$$T_n^0(q) = \begin{bmatrix} R_n^0(q) & o_n^0(q) \\ 0 & 1 \end{bmatrix}, \quad q = [q_1, \ldots, q_n]$$

there are several ways to represent rotational matrix $R_n^0(q)$, e.g.

$$R = R(\phi, \theta, \psi) = R_{z,\phi} R_{y,\theta} R_{z,\psi}$$

with (ϕ, θ, ψ) being the Euler angles (ZYZ-parametrization).

If the robot moves $q = q(t)$, then $\omega_{0,n}^0(t)$ is defined by

$$\frac{d}{dt} R_n^0(q(t)) = S(\omega_{0,n}^0(t)) R_n^0(q(t))$$

Angular velocity is function of $\alpha(t) = (\phi(t), \theta(t), \psi(t))$, $\frac{d}{dt}\alpha(t)$

$$\omega_{0,n}^0(t) = B(\alpha(t)) \frac{d}{dt}\alpha(t)$$

Analytic Jacobian

The equation

$$\begin{bmatrix} v_n^0(t) \\ \alpha(t) \end{bmatrix} = J_a(q)\dot{q}$$

defines the so-called analytic Jacobian

Analytic Jacobian

The equation

$$\begin{bmatrix} v_n^0(t) \\ \alpha(t) \end{bmatrix} = J_a(q)\dot{q}$$

defines the so-called analytic Jacobian

Both manipulator and analytic Jacobians are closely related

$$\begin{bmatrix} v_n^0(t) \\ \omega_{0,n}^0(t) \end{bmatrix} = J(q(t))\,\dot{q}(t) = \begin{bmatrix} v_n^0(t) \\ B(\alpha(t))\dot{\alpha}(t) \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ 0 & B(\alpha(t)) \end{bmatrix} \begin{bmatrix} v_n^0(t) \\ \alpha(t) \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ 0 & B(\alpha(t)) \end{bmatrix} J_a(q(t))\,\dot{q}(t)$$

Lecture 7: Kinematics: Velocity Kinematics - the Jacobian

- Manipulator Jacobian
- Analytical Jacobian
- Singularities
- Inverse Velocity and Manipulability

Singularities

The manipulator Jacobian $J(q)$ a $6 \times n$-matrix mapping

$$\xi = \begin{bmatrix} v_n^0 \\ \omega_{0,n}^0 \end{bmatrix} = J(q)\,\frac{d}{dt}q = \begin{bmatrix} J_v(q) \\ J_\omega(q) \end{bmatrix} \frac{d}{dt}q$$

$$= \left[J_1(q),\, J_2(q),\, \ldots,\, J_n(q)\right] \begin{bmatrix} \dot q_1 \\ \vdots \\ \dot q_n \end{bmatrix}$$

Singularities

The manipulator Jacobian $J(q)$ a $6 \times n$-matrix mapping

$$\xi = \begin{bmatrix} v_n^0 \\ \omega_{0,n}^0 \end{bmatrix} = J(q)\,\frac{d}{dt}q = \begin{bmatrix} J_v(q) \\ J_\omega(q) \end{bmatrix} \frac{d}{dt}q$$

$$= \left[J_1(q),\, J_2(q),\, \ldots,\, J_n(q)\right] \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

Since $\xi \in \mathbb{R}^6 \quad \Rightarrow \quad$ rank $\left[J_1(q),\, J_2(q),\, \ldots,\, J_n(q)\right] \leq 6, \ \forall\, q$

Singularities

The manipulator Jacobian $J(q)$ a $6 \times n$-matrix mapping

$$\xi = \begin{bmatrix} v_n^0 \\ \omega_{0,n}^0 \end{bmatrix} = J(q)\,\frac{d}{dt}q = \begin{bmatrix} J_v(q) \\ J_\omega(q) \end{bmatrix} \frac{d}{dt}q$$

$$= \left[J_1(q),\, J_2(q),\, \ldots,\, J_n(q)\right] \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

Since $\xi \in \mathbb{R}^6 \quad \Rightarrow \quad$ rank $\left[J_1(q),\, J_2(q),\, \ldots,\, J_n(q)\right] \leq 6,\ \forall\, q$

Configurations $q^s = \left[q_1^s,\, \ldots,\, q_n^s\right]$ for which

$$\text{rank } \left[J_1(q_s),\, J_2(q_s),\, \ldots,\, J_n(q_s)\right] < \max_q \left\{\text{rank } \left[J_1(q),\, J_2(q),\, \ldots,\, J_n(q)\right]\right\}$$

are called singular

Singularities

Identifying manipulator singularities are important:

- Singularities represent configurations from which certain direction may be unattainable;

Singularities

Identifying manipulator singularities are important:

- Singularities represent configurations from which certain direction may be unattainable;
- At singularity bounded end-effector velocity ξ may correspond to unbounded joint velocities $\dot{q}$

$$\xi_1 = J_1(q_1, q_2)\dot{q}_1 + J_2(q_1, q_2)\dot{q}_2$$

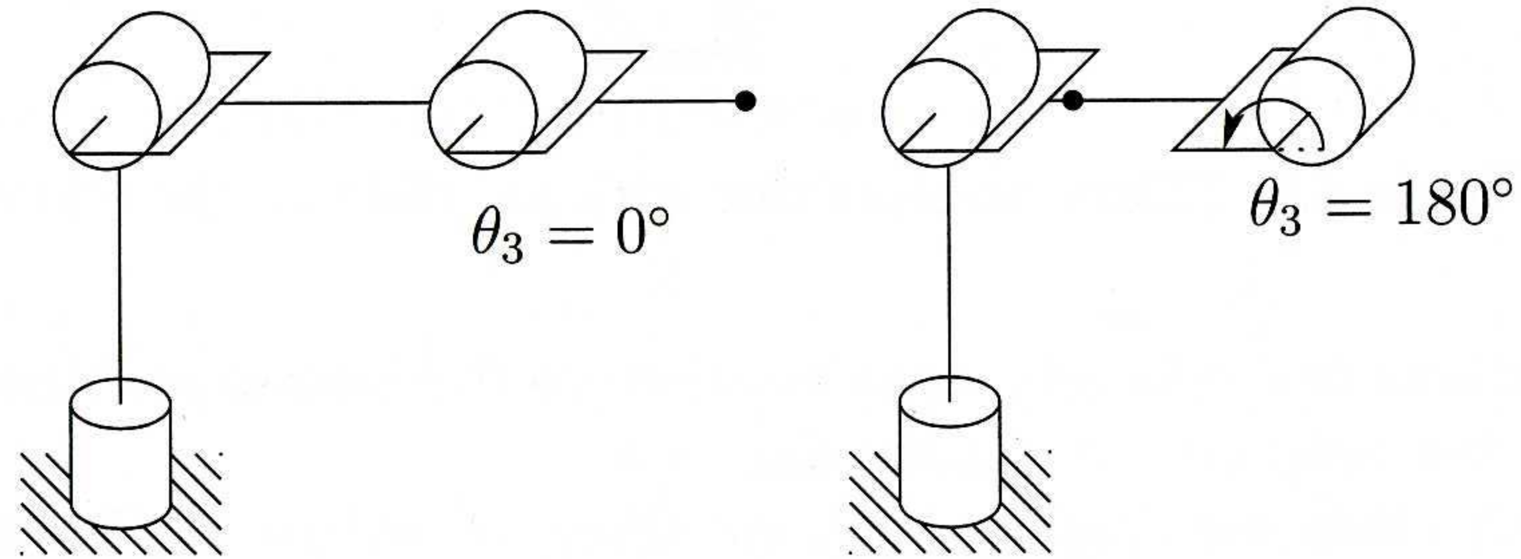
Singularities

Identifying manipulator singularities are important:

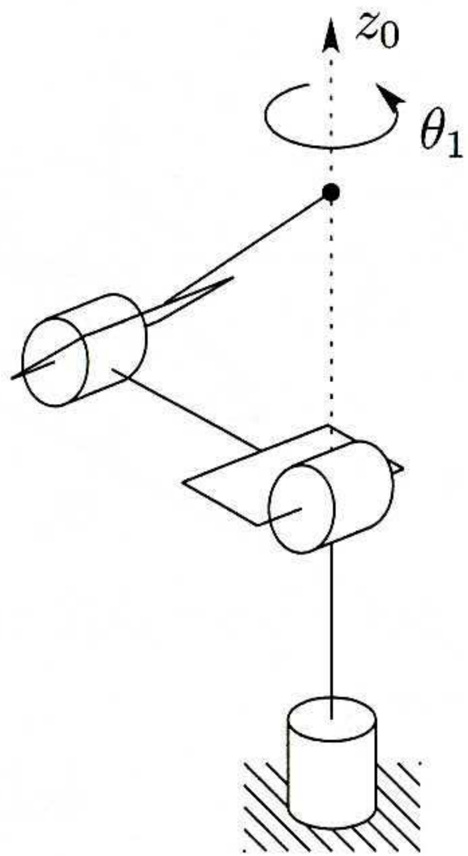
- Singularities represent configurations from which certain direction may be unattainable;
- At singularity bounded end-effector velocity ξ may correspond to unbounded joint velocities $\dot{q}$

$$\xi_1 = J_1(q_1, q_2)\dot{q}_1 + J_2(q_1, q_2)\dot{q}_2$$

- Singularities often correspond to points on the boundary of the manipulator workspace



Singularities of the elbow manipulator



Singularities of the elbow manipulator

Lecture 7: Kinematics: Velocity Kinematics - the Jacobian

- Manipulator Jacobian
- Analytical Jacobian
- Singularities
- Inverse Velocity and Manipulability

Assigning Joint Velocities

The manipulator Jacobian $J(q)$ a $6 \times n$-matrix mapping

$$\xi = \begin{bmatrix} v_n^0 \\ \omega_{0,n}^0 \end{bmatrix} = J(q)\,\frac{d}{dt}q = \left[J_1(q),\, J_2(q),\, \ldots,\, J_n(q)\right] \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

Assigning Joint Velocities

The manipulator Jacobian $J(q)$ a $6 \times n$-matrix mapping

$$\xi = \begin{bmatrix} v_n^0 \\ \omega_{0,n}^0 \end{bmatrix} = J(q)\,\frac{d}{dt}q = \left[J_1(q),\, J_2(q),\, \ldots,\, J_n(q)\right] \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

When the Jacobian is square, we can invert it

$$\dot{q} = J^{-1}(q)\xi_{des}$$

Assigning Joint Velocities

The manipulator Jacobian $J(q)$ a $6 \times n$-matrix mapping

$$\xi \;=\; \begin{bmatrix} v_n^0 \\ \\ \omega_{0,n}^0 \end{bmatrix} = J(q)\,\frac{d}{dt}q = \left[J_1(q),\; J_2(q),\; \ldots,\; J_n(q)\right] \begin{bmatrix} \dot q_1 \\ \vdots \\ \dot q_n \end{bmatrix}$$

When the Jacobian is square, we can invert it

$$\dot q = J^{-1}(q)\xi_{des}$$

When $n > 6$ there are many solutions, but them can be found

$$\dot q = J^{+}(q)\xi_{des} + \left(I - J^{+}J\right) b, \quad \forall\, b \in \mathbb{R}^n$$

where $J^{+} = J^{T}\left(JJ^{T}\right)^{-1}$

Manipulability

Suppose that we restricted possible joint velocities of n-degree of freedom robot

$$\|\dot{q}\|^2 = \|\dot{q}_1\|^2 + \cdots + \|\dot{q}_n\|^2 \leq 1$$

Manipulability

Suppose that we restricted possible joint velocities of n-degree of freedom robot

$$\|\dot{q}\|^2 = \|\dot{q}_1\|^2 + \cdots + \|\dot{q}_n\|^2 \leq 1$$

Then we restricted possible body velocities by

$$\xi^T P(q_a)\xi \; = \; \xi^T \left(J(q_a)J^T(q_a)\right)^{-1} \xi = \left(J^+(q_a)\xi\right)^T J^+(q_a)\xi$$

$$= \; \|\dot{q}\|^2 \leq 1$$

Manipulability

Suppose that we restricted possible joint velocities of n-degree of freedom robot

$$\|\dot{q}\|^2 = \|\dot{q}_1\|^2 + \cdots + \|\dot{q}_n\|^2 \leq 1$$

Then we restricted possible body velocities by

$$\xi^T P(q_a)\xi = \xi^T \left(J(q_a)J^T(q_a)\right)^{-1} \xi = \left(J^+(q_a)\xi\right)^T J^+(q_a)\xi$$

$$= \|\dot{q}\|^2 \leq 1$$

The set defined by the inequality is called manipulability ellipsoid

It illustrates how easy/difficult to move the end-effector

in certain directions from the configuration q_a