Lecture 3: Kinematics:

Rigid Motions and Homogeneous Transformations

- Composition of Rotations:
  - Rotations with Respect to the Current Frame
  - Rotations with Respect to the Fixed Frame
  - Example
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• Parameterizations of Rotations:
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  ◦ Roll, Pitch and Yaw Angles
  ◦ Axis/Angle Representation
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  ○ Axis/Angle Representation

• Rigid Motions and Homogeneous Transformations
What Have We Learned:

Given $N$-frames in the 3-dimensional space

$$(o_0, x_0, y_0, z_0), \ (o_1, x_1, y_1, z_1), \ \cdots \ (o_{N-1}, x_{N-1}, y_{N-1}, z_{N-1})$$
What Have We Learned:

Given $N$-frames in the 3-dimensional space

$$(o_0, x_0, y_0, z_0), \ (o_1, x_1, y_1, z_1), \ldots \ (o_{N-1}, x_{N-1}, y_{N-1}, z_{N-1})$$

If we are given $(N - 1)$-rotation matrices

$$R_0^0, \ R_1^1, \ldots, \ R_{N-1}^{N-2}$$

that represent consecutive rotation between the current frames

$$\{(x_0y_0z_0), (x_1y_1z_1)\}, \ \{(x_1y_1z_1), (x_2y_2z_2)\}, \ldots,$$

$$\{(x_{N-2}y_{N-2}z_{N-2}), (x_{N-1}y_{N-1}z_{N-1})\}$$
What Have We Learned:

Given $N$-frames in the 3-dimensional space

$$(o_0, x_0, y_0, z_0), \ (o_1, x_1, y_1, z_1), \ldots \ (o_{N-1}, x_{N-1}, y_{N-1}, z_{N-1})$$

If we are given $(N-1)$-rotation matrices

$$R_1^0, \ R_2^1, \ldots, \ R_{(N-1)}^{(N-2)}$$

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$$\{(x_0y_0z_0), (x_1y_1z_1)\}, \ \{(x_1y_1z_1), (x_2y_2z_2)\}, \ldots,$$

$$\{(x_{N-2}y_{N-2}z_{N-2}), (x_{N-1}y_{N-1}z_{N-1})\}$$

The formula to compute the position of the point in the 0-frame having known its position in the 1-frame

$$p^0 = R_1^0 p^1$$
What Have We Learned:

Given \( N \)-frames in the 3-dimensional space

\[
(o_0, x_0, y_0, z_0), \quad (o_1, x_1, y_1, z_1), \ldots (o_{N-1}, x_{N-1}, y_{N-1}, z_{N-1})
\]

If we are given \((N - 1)\)-rotation matrices

\[
R^0_1, \quad R^1_2, \quad \ldots, \quad R^{(N-2)}_{(N-1)}
\]

that represent consecutive rotations between the current frames

\[
\{(x_0y_0z_0), (x_1y_1z_1)\}, \quad \{(x_1y_1z_1), (x_2y_2z_2)\}, \ldots,
\]

\[
\{(x_{N-2}y_{N-2}z_{N-2}), (x_{N-1}y_{N-1}z_{N-1})\}
\]

The formula to compute the position of the point in the 0-frame
having known its position in the 2-frame

\[
p^0 = R^0_1p^1, \quad p^1 = R^1_2p^2
\]
What Have We Learned:

Given $N$-frames in the 3-dimensional space

$$(o_0, x_0, y_0, z_0), \ (o_1, x_1, y_1, z_1), \ldots \ (o_{N-1}, x_{N-1}, y_{N-1}, z_{N-1})$$

If we are given $(N - 1)$-rotation matrices

$$R_1^0, \ R_2^1, \ldots, \ R_{(N-1)}^{(N-2)}$$

that represent consecutive rotations between the current frames

$$\{(x_0y_0z_0), (x_1y_1z_1)\}, \ \{(x_1y_1z_1), (x_2y_2z_2)\}, \ldots,$$

$$\{(x_{N-2}y_{N-2}z_{N-2}), (x_{N-1}y_{N-1}z_{N-1})\}$$

The formula to compute the position of the point in the 0-frame having known its position in the $(N - 1)$-frame

$$p^0 = R_1^0p^1, \ p^1 = R_2^1p^2, \ldots, \ p^{(N-2)} = R_{(N-1)}^{(N-2)}p^{(N-1)}$$
What Have We Learned:

Given \( N \)-frames in the 3-dimensional space

\[
(o_0, x_0, y_0, z_0), \quad (o_1, x_1, y_1, z_1), \ldots \quad (o_{N-1}, x_{N-1}, y_{N-1}, z_{N-1})
\]

If we are given \((N - 1)\)-rotation matrices

\[
R_1^0, \quad R_2^1, \quad \ldots, \quad R_{(N-2)}^{(N-1)}
\]

that represent consecutive rotations between the current frames

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\{(x_0y_0z_0), (x_1y_1z_1)\}, \quad \{(x_1y_1z_1), (x_2y_2z_2)\}, \quad \ldots,
\]

\[
\{(x_{N-2}y_{N-2}z_{N-2}), (x_{N-1}y_{N-1}z_{N-1})\}
\]

The formula to compute the position of the point in the 0-frame having known its position in the \((N - 1)\)-frame is

\[
p^0 = R_1^0 R_2^1 R_3^2 \cdots R_{(N-2)}^{(N-1)} p^{(N-1)}
\]
Example 2.8:

Find the rotation $R$ defined by the following basic rotations:

1. A rotation of $\theta$ about the current axis $x$;
2. A rotation of $\phi$ about the current axis $z$.
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- 1: A rotation of $\theta$ about the current axis $\mathbf{x}$;
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The total rotation will be then

$$\mathbf{R} = \mathbf{R}_{\mathbf{x},\theta} \mathbf{R}_{\mathbf{z},\phi}$$
Example 2.8:

Find the rotation $R$ defined by the following basic rotations:

• 1: A rotation of $\theta$ about the current axis $x$;
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The total rotation will be then

$$R = R_{x,\theta} R_{z,\phi}$$

For any point of the 2-frame with coordinates $p^2 = [x^2, y^2, z^2]^T$ its coordinates in the 0-frame are computed simply as

$$p^0 = R p^2 = \left[R_{x,\theta} R_{z,\phi}\right] p^2$$
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The total rotation will be then

$$R = R_{x,\theta} R_{z,\phi} R_3$$
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The total rotation will be then

$$\mathbf{R} = \mathbf{R}_{x,\theta} \mathbf{R}_{z,\phi} \mathbf{R}_3$$

We have computed this rotation as

$$\mathbf{R}_3 = [\mathbf{R}_{x,\theta} \mathbf{R}_{z,\phi}]^{-1} \cdot \mathbf{R}_{z,\alpha} \cdot [\mathbf{R}_{x,\theta} \mathbf{R}_{z,\phi}]$$
Example 2.8:

Find the rotation $\mathbf{R}$ defined by the following basic rotations:

- 1: A rotation of $\theta$ about the current axis $x$;
- 2: A rotation of $\phi$ about the current axis $z$
- 3: A rotation of $\alpha$ about the fixed axis $z$

The total rotation will be then

$$
\mathbf{R} = \mathbf{R}_{x,\theta} \mathbf{R}_{z,\phi} \mathbf{R}_3 = \mathbf{R}_{z,\alpha} \mathbf{R}_{x,\theta} \mathbf{R}_{z,\phi}
$$

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\mathbf{R}_3 = \left[ \mathbf{R}_{x,\theta} \mathbf{R}_{z,\phi} \right]^{-1} \cdot \mathbf{R}_{z,\alpha} \cdot \left[ \mathbf{R}_{x,\theta} \mathbf{R}_{z,\phi} \right]
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- 1: A rotation of $\theta$ about the current axis $x$;
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The total rotation will be then

$$R = R_{x,\theta} R_{z,\phi} R_3 R_{y,\beta} = \left[ R_{z,\alpha} R_{x,\theta} R_{z,\phi} \right] R_{y,\beta}$$
Example 2.8:

Find the rotation $R$ defined by the following basic rotations:

1. A rotation of $\theta$ about the current axis $x$;
2. A rotation of $\phi$ about the current axis $z$;
3. A rotation of $\alpha$ about the fixed axis $z$;
4. A rotation of $\beta$ about the current axis $y$;
5. A rotation of $\delta$ about the fixed axis $x$.
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1. A rotation of $\theta$ about the current axis $x$;
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4. A rotation of $\beta$ about the current axis $y$;
5. A rotation of $\delta$ about the fixed axis $x$.

The total rotation will be then

$$R = R_{z,\alpha} R_{x,\theta} R_{z,\phi} R_{y,\beta} R_5$$
Example 2.8:

Find the rotation $R$ defined by the following basic rotations:

• 1: A rotation of $\theta$ about the current axis $x$;
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• 3: A rotation of $\alpha$ about the fixed axis $z$
• 4: A rotation of $\beta$ about the current axis $y$
• 5: A rotation of $\delta$ about the fixed axis $x$

The total rotation will be then

$$R = R_{z,\alpha} R_{x,\theta} R_{z,\phi} R_{y,\beta} R_5$$

We have computed this rotation as

$$R_5 = \left[ R_{z,\alpha} R_{x,\theta} R_{z,\phi} R_{y,\beta} \right]^{-1} \cdot R_{x,\delta} \cdot \left[ R_{z,\alpha} R_{x,\theta} R_{z,\phi} R_{y,\beta} \right]$$
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Find the rotation $R$ defined by the following basic rotations:

- 1: A rotation of $\theta$ about the current axis $x$;
- 2: A rotation of $\phi$ about the current axis $z$
- 3: A rotation of $\alpha$ about the fixed axis $z$
- 4: A rotation of $\beta$ about the current axis $y$
- 5: A rotation of $\delta$ about the fixed axis $x$

The total rotation will be then

$$R = R_{x,\delta} \cdot R_{z,\alpha} \cdot R_{x,\theta} \cdot R_{z,\phi} \cdot R_{y,\beta}$$

We have computed this rotation as

$$R_5 = \left[ R_{z,\alpha} \ R_{x,\theta} \ R_{z,\phi} \ R_{y,\beta} \right]^{-1} \cdot R_{x,\delta} \cdot \left[ R_{z,\alpha} \ R_{x,\theta} \ R_{z,\phi} \ R_{y,\beta} \right]$$
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Parameterizations of Rotations:

Any rotation matrix $R$ is

- of dimension $3 \times 3$, i.e. it is 9-numbers
- belongs to $SO(3)$, i.e.
  - its 3 columns are vectors of length 1 (3 equations)
  - its 3 columns are orthogonal to each other (3 equations)
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Any rotation matrix $R$ is

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- belongs to $SO(3)$, i.e.
  - its 3 columns are vectors of length 1 (3 equations)
  - its 3 columns are orthogonal to each other (3 equations)

Except the particular cases, only 3 of 9 numbers that parameterize the rotation matrix, can be assigned freely!
Euler Angles:

Euler angles are angles of 3 rotations about current axes

$$R_{ZYZ} := R_{z, \phi} \cdot R_{y, \theta} \cdot R_{z, \psi}$$
Euler Angles:

Euler angles are angles of 3 rotations about current axes

\[ R_{ZYZ} := \begin{bmatrix}
  c_\phi & -s_\phi & 0 \\
  s_\phi & c_\phi & 0 \\
  0 & 0 & 1
\end{bmatrix} \cdot R_{y,\theta} \cdot R_{z,\psi} \]
Euler angles are angles of 3 rotations about current axes

\[
R_{ZYZ} := \begin{bmatrix}
    c_\phi & -s_\phi & 0 \\
    s_\phi & c_\phi & 0 \\
    0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
    c_\theta & 0 & s_\theta \\
    0 & 1 & 0 \\
    -s_\theta & 0 & c_\theta
\end{bmatrix} \cdot R_{z, \psi}
\]
Euler Angles:

Euler angles are angles of 3 rotations about current axes

\[
R_{ZYX} := \begin{bmatrix}
    c_\phi & -s_\phi & 0 \\
    s_\phi & c_\phi & 0 \\
    0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
    c_\theta & 0 & s_\theta \\
    0 & 1 & 0 \\
    -s_\theta & 0 & c_\theta
\end{bmatrix} \cdot \begin{bmatrix}
    c_\psi & -s_\psi & 0 \\
    s_\psi & c_\psi & 0 \\
    0 & 0 & 1
\end{bmatrix}
\]
Roll, Pitch Yaw Angles:

Roll, pitch and yaw angles are angles of 3 rotations about the fixed axes $x$, $y$ and $z$

$$R_{xyz} := R_{z, \phi} \cdot R_{y, \theta} \cdot R_{x, \psi}$$
Roll, pitch and yaw angles are angles of 3 rotations about the fixed axes $x$, $y$ and $z$

$$R_{xyz} := R_{z,\phi} \cdot R_{y,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix}$$
Roll, Pitch Yaw Angles:

Roll, pitch and yaw angles are angles of 3 rotations about the fixed axes $x$, $y$ and $z$

$$R_{xyz} := R_{z, \phi} \cdot \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix}$$
Roll, Pitch Yaw Angles:

Roll, pitch and yaw angles are angles of 3 rotations about the fixed axes $x$, $y$ and $z$

$$R_{xyz} := \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix}$$
Axis/Angle Representation for a Rotation Matrix:

Any rotational matrix can be expressed as a rotation of angle $\theta$ about an axis $k = [k_x, k_y, k_z]^T$

$$R_{k,\theta} = R \cdot R_{z,\theta} \cdot R^{-1}, \quad R = R_{z,\alpha} \cdot R_{y,\beta}$$
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• Rigid Motions and Homogeneous Transformations
Rigid Motions

A rigid motion is an ordered pair $(R, d)$, where $R \in SO(3)$ and $d \in \mathbb{R}^3$. The group of all rigid motions is known as Special Euclidean Group denoted by $SE(3)$. 
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It is the way to compute coordinates in different frames

\[ p^0 = R_1^0 p^1 + d^0 \]
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It is the way to compute coordinates in different frames

\[
p^0 = R_1^0 p^1 + d^0
\]

If there are 3 frames corresponding to 2 rigid motions

\[
p^1 = R_2^1 p^2 + d_2^1
\]

\[
p^0 = R_1^0 p^1 + d_1^0
\]

then the overall motion is

\[
p^0 = R_1^0 R_2^1 p^2 + R_1^0 d_2^1 + d_1^0
\]
Concept of Homogeneous Transformation

HT is just a convenient way to write the formula

\[ p^0 = R^0_1 R^1_2 p^2 + R^0_1 d^1 + d^0 \]
Concept of Homogeneous Transformation

HT is just a convenient way to write the formula

\[ p^0 = R_1^0 R_2^1 p^2 + R_1^0 d_1^1 + d^0 \]

Given two rigid motions \((R_1^0, d_1^0)\) and \((R_2^1, d_2^1)\), consider the product of two matrices

\[
\begin{bmatrix}
R_1^0 & d_1^0 \\
0_{1 \times 3} & 1
\end{bmatrix}
\begin{bmatrix}
R_2^1 & d_2^1 \\
0_{1 \times 3} & 1
\end{bmatrix}
= 
\begin{bmatrix}
R_1^0 R_2^1 & R_1^0 d_2^1 + d_1^0 \\
0_{1 \times 3} & 1
\end{bmatrix}
\]
Concept of Homogeneous Transformation

HT is just a convenient way to write the formula

\[ p^0 = R_0^0 R_1^1 p^2 + R_0^0 d_1^1 + d_0^0 \]

Given two rigid motions \((R_0^0, d_0^0)\) and \((R_1^1, d_1^1)\), consider the product of two matrices

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Concept of Homogeneous Transformation

HT is just a convenient way to write the formula

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Given two rigid motions \((R_1^0, d_1^0)\) and \((R_2^1, d_2^1)\), consider the product of two matrices

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0_{1\times3} & 1
\end{bmatrix} =
\begin{bmatrix}
R_1^0 R_2^1 & R_1^0 d_2^1 + d_1^0 \\
0_{1\times3} & 1
\end{bmatrix}
\]

Given a rigid motion \((R, d) \in SE(3)\), the \(4 \times 4\)-matrix

\[ H = \begin{bmatrix}
R & d \\
0_{1\times3} & 1
\end{bmatrix} \]

is called homogeneous transformation associated with \((R, d)\)
Concept of Homogeneous Transformation

To use HTs in computing coordinates of points, we need to extend the vectors $p^0$ and $p^1$ by one coordinate. Namely

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix}, \quad P^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix}$$
Concept of Homogeneous Transformation

To use HTs in computing coordinates of points, we need to extend the vectors \( p^0 \) and \( p^1 \) by one coordinate. Namely

\[
P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix}, \quad P^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix}
\]

Then

\[
P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R^0 \ p^1 + d^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R^0 \\ 0_{1 \times 3} \\ H^0 \\ 1 \end{bmatrix} \begin{bmatrix} p^1 \\ 1 \end{bmatrix}
\]

that is in short

\[
P^0 = H^0 P^1
\]